LEVEL 3 FIELDS LINES AND GUIDES

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Chapter 1

INTRODUCTION

1.1 Introduction

1.1.1 Course extent

This course consists of 26 lectures delivered at the rate of one per week throughout the year. It forms a necessary theoretical background for two experiments in the Level 3 Experimental Electrical Engineering III and IIIC courses, for study by both Electrical Engineering and Computer Systems Engineering students in their final year, and for profesional practice.

1.1.2 Course aims

The aims of the course are

- To provide a restatement of the basic principles of electrodynamics which have been studied at Levels 1 and 2. Later in the course, to provide useful extensions to those principles.
- To review the concepts of *lumped circuit theory*, and to recognise the assumptions underlying that theory.
- To recognise that these assumptions are not valid in a wide range of electrical engineering contexts.
- To see how the laws of electrodynamics can be used to produce a version of circuit theory, known as *distributed circuit theory*, which is valid in a wider range of contexts than is lumped circuit theory.
- To exercise that distributed circuit theory to derive the important characteristics of transmission lines, particularly in a communications context.
- To recognise that there are communications contexts in which the assumptions of neither lumped circuit theory nor distributed circuit theory are valid.
- To show how *electromagnetic field theory* can be used to derive the characteristics of practically important transmission structures in those contexts. Included in those contexts is the *radiation situation*, in which charges and currents on a *transmitting*

antenna launch electromagnetic waves into free space, and where these waves induce charges and currents on a *receiving antenna*.

The program followed by the course is to pursue those aims approximately in the order listed.

1.1.3 Relevance

The material in this course is relevant to

- power engineers in their study of the transmission of power over long distances;
- communication engineers in their study of propagation of signals in a wide variety of media over both short and long distances;
- computer systems engineers in their study of the behaviour of high speed information bearing signals between and within computer systems; and
- information technologists in their study of the physical processes of information dissemination.

1.2 Function of this chapter

The functions of this chapter are

- To establish suitable *notation* by means of which the concepts of this course may be discussed.
- To provide revision of some fundamental concepts.
- To indicate why in many electrical engineering contexts the simple methods of the lumped circuit analysis courses studied at Level 1 and Level 2 do not provide a valid description.
- To give some indication of the practical uses of transmission lines and guided waves.

1.3 Notation

1.3.1 Scalar quantities

Much of the time in this course we will be dealing with variables which directly express the values of the physical quantities, such as, for example, voltage or current. If those physical quantities have a time variation, so do the variables of our equations.

In such a case, when the quantities represented are *scalars*, as in the example of voltage or current just mentioned, we use lower case Roman of sometimes Greek letters to represent them. Sometimes the time variation is shown, and sometimes it is not, as for example in the equation for a real time-varying voltage

$$v = v(t). \tag{1.1}$$

In many cases it will be convenient to restrict the time variation of all physical quantities to be either constant or *sinusoidal*, or more explicitly to be of *cosine* form.

In such cases, the behaviour of each time-varying quantity is known for all time if we know the *frequency*, the *amplitude* and the *phase* of the cosine function of time. In a single context, all such quantities are *assumed to have the same frequency*, which is stated once as a fixed part of that context, but the different variables representing different quantities can have *various amplitudes and phases.*

In this situation, it is convenient to introduce a variable called a *complex phasor* which while not itself being a function of time, does represent in the manner described below a time-varying quantity. What we do is to introduce a complex number, constructed so that the *magnitude* of the complex number is the *amplitude* of the cosine waveform, and the *angle* of the complex number in its polar representation, as shown for example in Figure 1.1, is the *phase angle* of the cosine waveform.

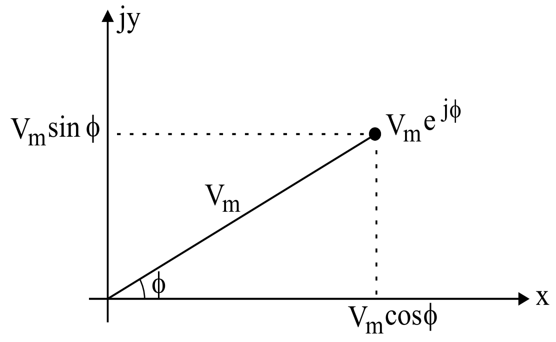


Figure 1.1: Polar representation of a complex number.

Thus for the sinusoidally varying quantity

$$v = V_m \cos(\omega t + \phi) \tag{1.2}$$

which has an amplitude V_m and a phase angle ϕ, we construct using V_m and ϕ the time invariant *complex phasor* V given by

$$\mathrm{V} = V_m e^{j\phi} \tag{1.3}$$

It may be noted that the relation between the *time invariant* complex phasor V and the *time-varying* variable $v(t)$ which it represents is

$$\boxed{v(t) = \Re\left\{\mathrm{V} e^{j\omega t}\right\}} \tag{1.4}$$

In words, this relation says that to recover the time function from the complex phasor, we multiply the complex phasor by $e^{j\omega t}$ and take the real part.

A *graphical* interpretation of the mathematical operation just defined is that to recover the time function from the complex phasor, we can first represent as shown in Figure 1.1

the phasor in the Argand diagram, and take its *projection* on the horizontal axis as the *expression of the value of the time function at the time* $t = 0$. To visualise the behaviour of the physical quantity *as a function of time*, we must construct a rotating arm whose position at $t = 0$ is that of the complex phasor, and rotate it in a counter-clockwise direction on the Argand diagram at an angular frequency ω, starting at time $t = 0$ at the position illustrated in Figure 1.1, and watch the values of the *projection on the horizontal axis* of the rotating arm which results.

Notice in the above exposition that we have *not* said that the *phasor* rotates. To do so would contradict the definition of the phasor as a *time invariant* quantity. We have given a different name, namely *rotating arm*, to the thing that rotates.

In establishing the notation described above, we have been able, because it is available, to use different calligraphy, namely v (italic) and V (upright Roman), to distinguish the *real time-varying* variables directly representing the physical quantities, and the *time invariant complex phasors* indirectly representing them. The difference in notation is helpful in avoiding misunderstandings.

1.3.2 Vector quantities

When we come to the representation of a sinusoidally varying *vector quantity*, we use as is common **bold face** characters to represent vectors. It is traditional to represent electromagnetic field quantities by upper case letters. If we are to distinguish clearly in our notation between the *real time-varying variables* and the *time invariant complex phasors* which in the particular case of sinusoidal time variation may be used to represent physical quantities, we are in need of two bold face upper case character sets of different appearance.

This need is satisfied by the use of so called *calligraphic characters*, for example E, D, H, and B, for the *time-varying* vectors directly representing time-varying physical vector quantities, and upright Roman letters, such as **E**, **D**, **H**, and **B**, for the *time invariant complex vectors* which can be used to represent sinusoidally varying vector quantities.

In both cases the term *vector* indicates that the physical quantity has in the three dimensional space in which we live three Cartesian components, each of which is a scalar quantity, and each of which may have any of: no time variation, a general non-sinusoidal time variation, or perhaps a sinusoidal variation. In all three cases representation by the calligraphic letters is appropriate, but only in the last case is representation in the phasor notation also appropriate.

The field vectors can in the general (non-sinusoidal) case have, in addition to their *time variation*, a *spatial variation*. In setting out equations, we may for emphasis explicitly show the time or space variation, or we may for compactness just write the symbol for the variable with the functional variation understood. Both of these things are done in the equation for an electric field vector

$$\mathsf{E} = \mathsf{E}(x, y, z, t). \tag{1.5}$$

Just as we have for scalar quantities the possibility of representing a sinusoidally varying quantity by a time invariant complex phasor, so we can for a sinusoidally varying vector quantity E introduce a time invariant complex vector phasor **E** (really just three phasors representing respectively the components along the three spatial co-ordinate axes), with the relation between the two being

$$\boxed{\mathsf{E}(x,y,z,t) = \Re\left\{\mathbf{E}(x,y,z)e^{j\omega t}\right\}} \tag{1.6}$$

which shows in the case of sinusoidal time variation of a vector the relation between the vector of *time-varying* functions providing a direct representation of the field E and the vector of *time invariant* complex phasors providing the indirect representation of that field.

1.3.3 Calligraphic characters

As some of the calligraphic characters are a little unusual in appearance, we provide here a table showing the most often used letters paired with their Roman counterparts. The table also, for future reference, gives the names and the Standard International Units of the physical quantities most commonly represented by those variables.

E	$\mathbf{E}$	Electric field intensity	Vm^{-1}
H	$\mathbf{H}$	Magnetic field intensity	Am^{-1}
D	$\mathbf{D}$	Electric flux density	Cm^{-2}
B	$\mathbf{B}$	Magnetic flux density	Wbm^{-2}
J	$\mathbf{J}$	Volume current density	Am^{-2}

Table 1.1: Some calligraphic characters.

1.4 Charge Conservation Concepts

As our first major concept we begin with the notation that there is a thing called *charge* which is *conserved*. It can be stationary or in motion. It can be used to probe an electromagnetic field; it is discrete but in such small amounts as not to concern us at the macroscopic level. The charge and its motion are characterised by parameters

1.4.1 Descriptors of charge and current

DESCRIPTOR	SYMBOL	UNITS
Charge density per unit volume	ρ	Cm^{-3}
Volume current density	J	Am^{-2}
Surface current density	K	Am^{-1}

Table 1.2: Important charge and current density descriptors.

which are illustrated in Figure 1.2 in a way such that each of the expressions in that Figure gives the amount of charge per unit time, i.e the current, crossing the indicated boundary.

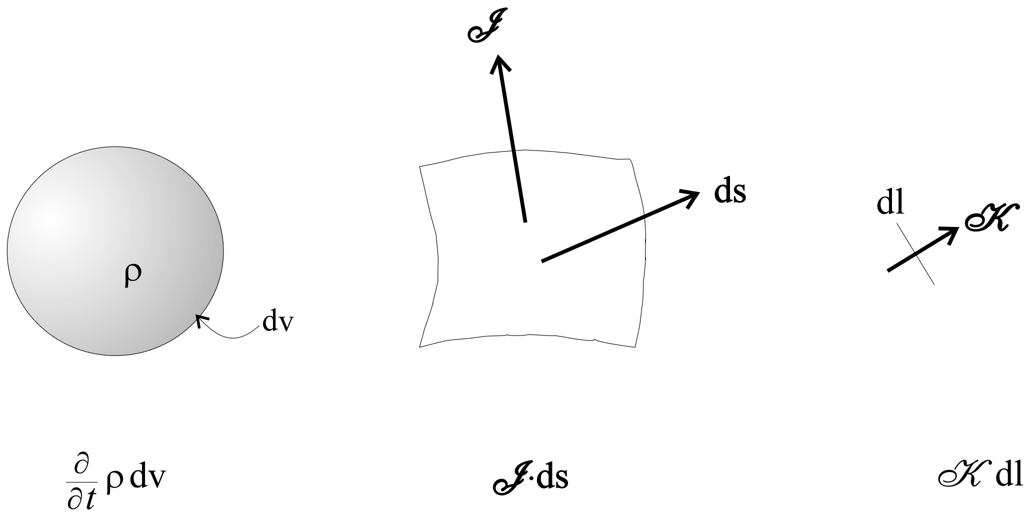


Figure 1.2: Illustration of charge and its motion.

Students should not proceed further in this course until they have a firm grasp of these concepts, and of the reasons why the names of the concepts sometimes belie the associated units.

1.4.2 Charge conservation equation

The equations expressing the fact that charge cannot be created or destroyed, but can merely be moved around in the form of an electric current, which if of suitable non-uniformity in space might cause a time rate of change of charge density to arise, is in integral form

$$\boxed{\oint_S \mathbf{J} \cdot \mathbf{ds} = -\frac{\partial}{\partial t}\int_v \rho dv} \tag{1.7}$$

or in differential form

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}} \tag{1.8}$$

The derivation of the differential form from the integral form is via Gauss' law of the vector calculus.

1.5 The Field and Material Vectors

The names, usual symbols, and Standard International units for the four vectors used to describe electromagnetic fields in a general medium are

E	electric field intensity	Vm^{-1}
H	magnetic field intensity	Am^{-1}
D	electric flux density	Cm^{-2}
B	magnetic flux density	Wbm^{-2}

The units of magnetic flux density B have the alternative name of Tesla, for which the abbreviation is T.

The names, usual symbols, and Standard International units for the vectors used to describe the state of a dielectric medium and the state of a magnetic medium respectively are

P	polarisation	Cm^{-2}
M	magnetisation	Am^{-1}

The fully general relations between the above six vectors are

(i)	$\mathsf{D} = \epsilon_0 \mathsf{E} + \mathsf{P}$	by definition
(ii)	$\mathsf{B} = \mu_0(\mathsf{H} + \mathsf{M})$	by definition

The values of the magnetic permeability and the dielectric permittivity of free space are in Standard International units

(i)	μ_0	$4\,\pi \times 10^{-7}$	Hm^{-1}	by definition
(ii)	ϵ_0	8.854×10^{-12}	Fm^{-1}	approximately

1.6 The Laws of Electrodynamics

We provide for reference below compact but complete statements of the four fundamental laws of electrodynamics which are embodied in what are known as Maxwell's equations. In the form stated below the laws are of full generality, and apply both to empty space and to regions containing material media, with those media being capable of either linear or non-linear behaviour.

1.6.1 Faraday's law

The circulation of the electric field vector E around a closed contour is equal to minus the time rate of change of the magnetic flux passing through a surface bounded by that contour, the positive direction of the surface being related to the positive direction of the contour by the right hand rule. In a mathematical formula this law takes the form

$$\boxed{\oint_C \mathsf{E} \cdot \mathbf{dr} = -\frac{d}{dt}\int_S \mathsf{B} \cdot \mathbf{ds}} \tag{1.9}$$

1.6.2 Ampere's law as modified by Maxwell

The circulation of the magnetic field vector H around a closed contour is equal to the sum of the conduction current and the displacement current passing through a surface bounded by that contour, with again the right hand rule relating the senses of the contour and the surface. In a mathematical formula this law takes the form

$$\boxed{\oint_C \mathsf{H} \cdot \mathbf{dr} = \int_S \mathsf{J} \cdot \mathbf{ds} + \frac{d}{dt} \int_S \mathsf{D} \cdot \mathbf{ds}} \tag{1.10}$$

1.6.3 Gauss' law for the electric flux

The total electric flux (defined in terms of the D vector) emerging from a closed surface is equal to the total *conduction charge* contained within the volume bounded by that surface. In a mathematical formula this law takes the form

$$\boxed{\oint_S \mathsf{D} \cdot \mathbf{ds} = \int_v \rho dv} \tag{1.11}$$

In the above equation we are expected to remember that the charge density appearing on the right hand side contains only the *conduction* charge density ρ^c.

1.6.4 Gauss' Law for the magnetic flux

The total magnetic flux (defined in terms of the B vector) emerging from any closed surface is zero. In a mathematical formula this law takes the form

$$\boxed{\oint_S \mathsf{B} \cdot \mathbf{ds} = 0} \tag{1.12}$$

1.7 Properties of Vector Fields

1.7.1 Source and vortex fields

In our study of electric and magnetic fields we will be greatly aided by the concepts of *sources* and *vortices*, originally developed, as the names suggest, in the study of fluid dynamics. The concepts themselves will be explained in the next section, but we will remark first that

- An ability to form in the mind electromagnetic field pictures appears to be an essential skill required for mastering electromagnetic theory.

- The source and vortex concepts provide a basis for picturing the electromagnetic field created in a wide range of situations.

- In a theorem first proved by Helmholtz, it may be shown that any vector field may be uniquely decomposed as the sum of two vector fields, one of which is purely *source type*, and one of which is purely *vortex type*.

- The sources and vortices of a field are described in a mathematical sense by derivatives of the vector calculus known as the *divergence* and *curl* derivatives.

- Maxwell's equations (which are the fundamental laws of electrodynamics) are when placed in differential form direct statements about the source and vortex properties of electromagnetic fields.

1.7.2 Source-type fields

Within a small region, a field is considered to be *source-type* if there is, as illustrated in Figure 1.3, a net flux of the field which emerges from that region.

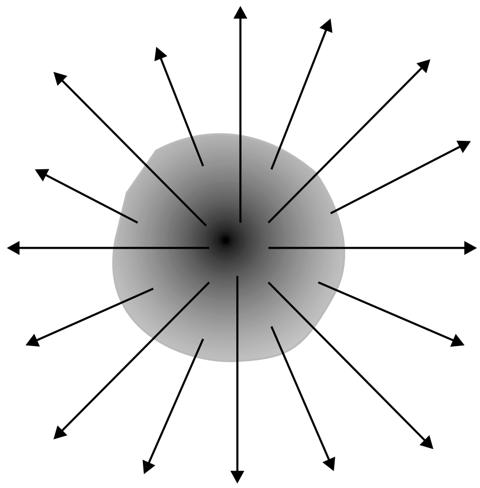


Figure 1.3: Illustration of a source type field.

The archetypical example of a source-type field is the *electrostatic field* which is caused by an unvarying distribution of electric charge within the field region.

1.7.3 Vortex-type fields

Within a small region, a field is considered to be *vortex-type* if there is, as illustrated in Figure 1.4, a net *circulation* of the field around a contour within that region.

The archetypical example of a vortex-type field is the *magnetostatic field* which is caused by an unvarying distribution of electric current within the region.

1.7.4 Gauss' theorem

Consider a closed surface S, of which $\mathbf{ds}$, *sensed outward,* is a vector element of surface area, enclosing a volume v, as shown in Figure 1.5. If $\mathbf{F}$ is an arbitrary vector field, it has been shown by Gauss that

$$\boxed{\oint_S \mathbf{F} \cdot \mathbf{ds} = \int_v \operatorname{div} \mathbf{F} \, dv.} \tag{1.13}$$

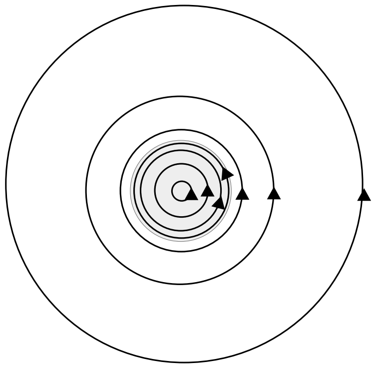


Figure 1.4: Illustration of a vortex type field.

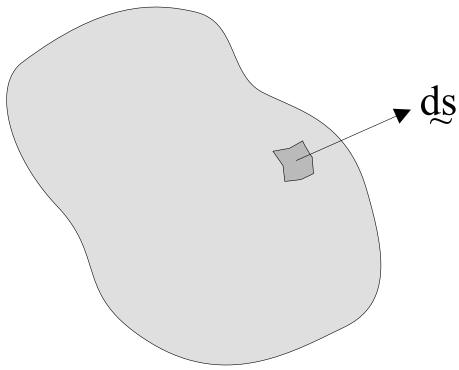


Figure 1.5: A closed surface for Gauss' theorem.

This theorem states in words that the net flux of a vector **F** emerging from a closed surface is the integral of the divergence of that vector over the volume enclosed by that surface.

1.7.5 Circulation of a Vector

When a closed path C is defined in a vector field **F** as shown Figure 1.6 we may define the *circulation* Γ of **F** around the contour C as

$$\Gamma = \oint_C \mathbf{F} \cdot \mathbf{dr} \tag{1.14}$$

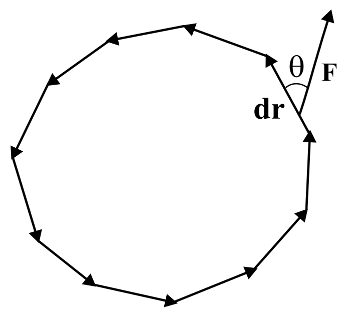


Figure 1.6: Circulation of a vector field around a contour.

It is clear that the circulation depends upon the shape and position of the contour in the field, and upon the *direction in which it is traversed* in the performance of the line integral. Thus it is necessary to define a *positive direction* for the contour. This has been done in Figure 1.6 by means of the arrows shown. Again, as with the definition of the positive direction for a surface, the positive direction may be arbitrarily chosen, but in the context of Stokes' theorem discussed below, a restriction applies.

1.7.6 Stokes' theorem

When a vector field intersects a non-closed surface S bounded by a contour C as shown in Figure 1.7, and the sense of the contour is chosen so that it is related to the sense of the surface area by the right hand rule, it has been shown by Stokes that

$$\boxed{\oint_C \mathbf{F} \cdot \mathbf{dr} = \int_S \mathbf{curl\,F} \cdot \mathbf{ds}} \tag{1.15}$$

Notice that the line integral is a *closed* one, but the surface integral is *not closed.* Stokes' theorem may be put into words as stating that the *circulation* of a vector field around a contour bounding a surface is equal to the *flux of the curl* of that field passing through that surface.

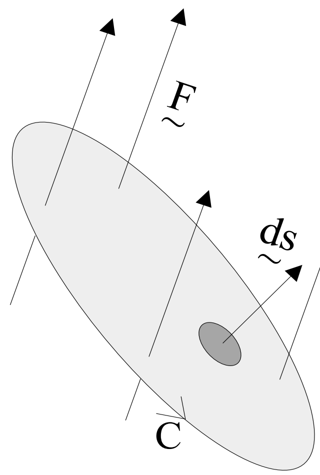


Figure 1.7: Vector field crossing a surface S.

1.8 Differential Forms of Electrodynamic Laws

With the aid of Gauss' and Stokes' theorems we may transform the laws of electrodynamics given in integral form in Section 1.6 to the equivalent differential forms below.

$$\boxed{\begin{aligned} \nabla \times \mathsf{E} &= -\frac{\partial \mathsf{B}}{\partial t} \\ \nabla \times \mathsf{H} &= \mathsf{J} + \frac{\partial \mathsf{D}}{\partial t} \\ \nabla \cdot \mathsf{D} &= \rho \\ \nabla \cdot \mathsf{B} &= 0 \end{aligned}} \tag{1.16}$$

These equations, together with the definitions

$$\boxed{\mathsf{D} = \epsilon_0 \mathsf{E} + \mathsf{P}} \tag{1.17}$$

and

$$\boxed{\mathsf{B} = \mu_0 (\mathsf{H} + \mathsf{M})} \tag{1.18}$$

are regarded as the *basic laws of electrodynamics* in the presence of media. Note that they are quite general in that they do not assume linearity or spatial uniformity of those media.

1.9 Lumped Circuit Theory

It is now appropriate to review the principles of that circuit theory, designated herein as *lumped circuit theory* as it has been studied at Levels 1 and 2, to recognise that the assumptions underlying that theory are rarely satisfied, and to appreciate the need for a form of circuit theory, to be known herein as *distributed circuit theory*, of wider applicability.

The picture which will emerge is that the complete laws of electrodynamics are as just stated in Maxwell's equations, whereas the forms of circuit analysis studied so far, and to which on account of their simplicity and apparent sufficiency we may have become attached, is merely an approximation with quite limited validity.

1.9.1 Laws of Lumped Circuit Theory

The formal exposition of lumped circuit theory is generally based on Kirchhoff's voltage and current laws, which are discussed in turn below, together with their underlying assumptions.

The discussion is illuminated by Figure 1.8. In this simple circuit an a.c. voltage generator is connected over wires of non-zero length and non-zero surface area to an inductor L and capacitor C. The two nodes of the circuit are labelled as N_1 and N_2. Points P, Q, R, S identify physical locations on some of the interconnecting wires.

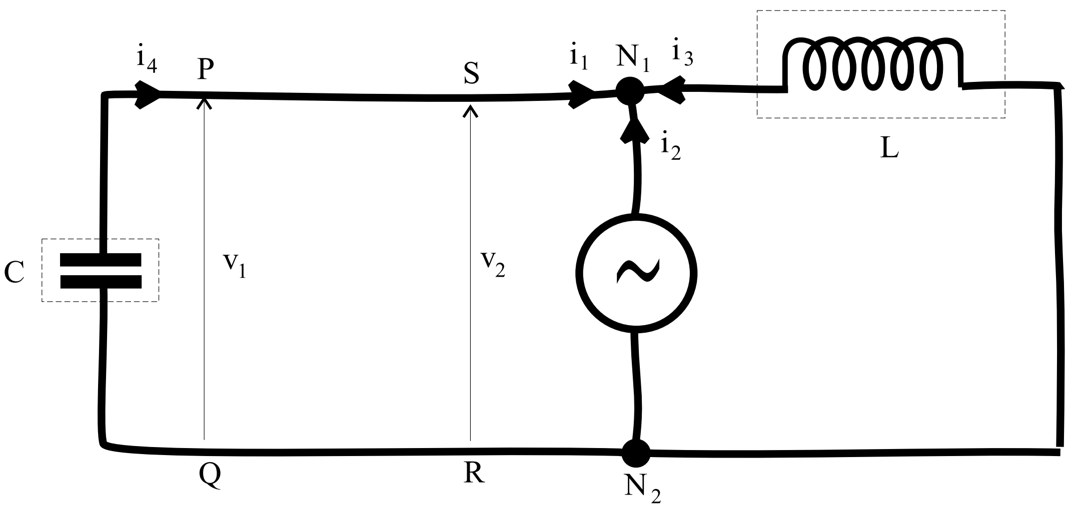


Figure 1.8: Some aspects of a a.c. circuit

Kirchhoff's voltage law

Kirchhoff's voltage law of lumped circuit theory states that the sum of the voltage rises in traversing any closed path in a circuit is zero. The voltage rises are of course differences in electric potential. The statement above is equivalent to the statement that the line integral of the electric field between any two points is independent of the path.

Now this last statement is clearly true of electrostatics, but is not generally true of electrodynamics, wherein changing magnetic fluxes can induce an emf around a closed path, and the electric field is no longer simply derivable as the gradient of a potential. So we may expect Kirchhoff's voltage law to apply to d.c. circuits, but we must be cautious in applying it to a.c. circuits.

The fact that lumped circuit theory is frequently applied to a.c. circuits rests upon a classification scheme at work in the background of our view of such circuits. In the lumped circuit view of an a.c. circuit, we divide the circuit into a number of regions. In some of those regions there are no changing magnetic fluxes, and within each of those limited regions the line integral of the electric field is independent of the path, as long as we remain within that region.

In some other regions, such as in the interior of transformers and inductors, and shown by the dotted area surrounding inductor L in Figure 1.8, we do allow changing magnetic fluxes, but in our consideration of the line integral of the electric field along a path from one terminal at one end of a winding of the device to a second terminal at the other end of that winding, we follow a *unique path*, namely along that winding, and thus always obtain the same value for the integral.

What we are doing in the above view is consigning the changing magnetic fluxes to limited regions of space, namely the interiors of inductors and transformers, and we are deliberately neglecting the effects of any magnetic fields which are caused by the currents on the wires which connect the separate devices. Clearly this is an approximation which becomes more and more questionable as frequency increases.

In the lumped theory approximation the line integral of the electric field from point Q to point P, which would have a value $-v_1$, ought to be equal to the line integral of electric field from point R to point S, which would have the value $-v_2$, but in reality the currents flowing on the interconnecting wires produce magnetic fluxes which have a component perpendicular to the page such that along the closed path PQRSP the integral of the electric field is by Faraday's law non-zero so that the variables v_1 and v_2 cannot be equal.

Kirchhoff's current law

Kirchhoff's current law states that the algebraic sum of the currents entering a single node of circuit is zero.

There are a number of assumptions underlying this statement. The first is that we know where the currents are in the sense that the currents flow on wires, we know where the wires are, and the current which flows into one end of a wire comes out the other. If we were to use bare wires, and dip our circuit into a conducting fluid such as sea water, none of this would be true - significant currents would leave the wires all along their length and significant current density would flow with some unknown distribution throughout the fluid.

Because living at the bottom of the ocean is for us uncomfortable, we do not commonly do this, but an equivalent thing is happening when we consider the effects of *displacement current* arising from changing electric fields between the wires. The electric flux density which originates at one wire must be supported by an equal surface charge density on that wire (whether it is insulated or not) and if the electric flux is changing in time, so will the surface charge density. Because of the principle of conservation of charge, this changing surface charge must come from somewhere, and it causes the current which flows along the wire to leave behind little bits of charge along the length of that wire so that the current entering one end of the wire is no longer equal to the current leaving at the other end. In relation to Figure 1.8 this would ensure that current i_4 entering the top-most wire is not equal to the current i_1 leaving it. For steady currents, however, these considerations do not apply.

Similar considerations apply at a circuit node - if the node is at a changing potential, the changing electric flux associated with that potential must be supported by a changing surface charge density, and if the node has a non-zero physical surface area, there is a changing charge thereon, and the algebraic sum of the currents entering that node is no longer exactly zero. In relation to Figure 1.8 this effect will guarantee that the sum of the currents i_1, i_2 and i_3 is not zero. For steady currents, however, these considerations do not apply.

The two effects discussed above lead to clear contradictions of Kirchhoff's current law.

As seen above in the discussion of Kirchhoff's voltage law, an assumption commonly made is that the magnetic fields of inductors are confined to just the region of the inductor as illustrated by the dotted line in Figure 1.8. Another assumption commonly made is that the electric fields associated with capacitors are confined to a limited region containing the capacitor as shown by the dotted line in Figure 1.8. Such an assumption underlies an assumption normally made in lumped circuit theory, that is that the charge on one plate of a capacitor is equal and opposite to the charge on the other.

This situation can be closely approximated in the normal construction of a capacitor where the plates are close together and have an extensive area and the electric field is confined largely to the region between those plates. But if the capacitor does not have this construction, and is opened out so that the plates have significant separation, the electric field which originates on one plate might terminate on parts of the circuit other than the other plate, and the assumption of equal and opposite charges on the two plates breaks down.

So we see that Kirchhoff's current law for a.c. circuits is an approximation, which depends on a number of factors including the geometry normally adopted for the construction of capacitors, and the validity of which diminishes as the frequency is raised.

Finally if we continue our survey of assumptions implicit in lumped circuit theory there is commonly made the assumption that magnetic fields H are induced just by conduction currents J via the magnetostatic relation

$$\nabla \times \mathsf{H} = \mathsf{J} \tag{1.19}$$

whereas, in reality the version of Ampere's law amended by Maxwell to

$$\nabla \times \mathsf{H} = \mathsf{J} + \frac{\partial \mathsf{D}}{\partial t} \tag{1.20}$$

indicates that magnetic fields can equally well be caused by *changing electric flux*, ie by the displacement currents.

1.10 Higher Level Theories

The effects of these departures from the assumptions of lumped circit theory become significant when the dimensions of the apparatus we are using become comparable with the electromagnetic wave length at the frequency of operation. In this situation we need a more advanced theory which makes fewer assumptions or simplifying approximations. Such advanced theories come in two levels of complexity discussed below.

1.10.1 Distributed circuit theory

The first of these is known as *distributed circuit theory* (which is one step up from lumped circuit theory) and resembles lumped circuit theory to some degree. This level of theory is applicable when we still have

- an idea of where the currents are, either on thin wires or spread uniformly over either the cross section or the surface of a wire of finite size, and
- some idea of the shape of the magnetic field distribution and that there are *planes in which although the magnetic field varies from point to point within the plane, the magnetic field is everywhere parallel to that plane.*

When this last condition is satisfied we have a situation where the line integral of the electric field between two points in that plane is *independent of the path* (provided the path is confined to that plane) and we can still usefully speak of the potential difference between two points in the plane.

In distributed circuit theory current and potential concepts may still be used subject to the limitations just recognised. We will apply this methodology extensively in our study of transmission lines in Chapters 2 and 3.

1.10.2 Electromagnetic field theory

There are, however, situations when the assumptions of neither distributed circuit theory nor lumped circuit theory are applicable. In these contexts we are obliged to deal directly in terms of the *electromagnetic fields*.

The need for field theory can come about for several reasons. One is that we are not sure just where the currents flow. They may still be on conductors, but the conductors are no longer filamentary, and we may have more current on one part of a conductor than another, and we are not sure by how much.

Secondly the magnetic flux density distribution may be such that it does not arrange itself in planes. Thirdly the displacement current may be comparable with the conduction current in the same region. There may even be no conduction current, and the magnetic field may be created purely by displacement current. This is exactly what happens in electromagnetic wave propagation.

When any of these things occurs we generally need the full field solution as discussed in Chapters 8 to 11.

1.10.3 Some aspects of field theory

Some interesting situations then arise. It may be impossible to find an unambiguous definition for either voltage or for current. We will always have the field vectors E, D, H, and B, but it may be that unambiguous definitions for voltage and current cannot be found.

1.11 How We Will Proceed

In the early chapters we will focus on the behaviour of *transmission lines* which will be frequently of the *twin line* or *coaxial line* structure. The lines are usually *uniform in cross section with respect to the longitudinal direction.* Magnetic fields will usually be completely *transverse*, and a distributed circuit treatment will be possible.

An important property of the transmission lines is that once you take into account the effects of distributed circuit theory, the current and voltage at one end of the line is seen to be not the same as the current and voltage at the other. Clearly in this context the wires of transmission line are not behaving as the conductors of a lumped circuit in which the potential has the same value all the way along the wire (except for the effect of resistance) and the current entering one end of the wire is the same as the current leaving.

Later in chapters 8 to 11 we will have to consider more complex geometries for signal transmission, in which situations magnetic fields in a longitudial direction make it necessary for us to use a more general treatment based upon electromagnetic field theory.

1.12 Uses of Transmission Lines

Transmission lines are commonly used for getting either power or signals in a regulated way from one point to another, without those signals radiating into space or becoming otherwise coupled to other circuits where they are not wanted.

Transmission lines are normally made *uniform in cross section.* One reason for making them uniform in cross section is that if the geometry is regular all of the effects discussed earlier become predictable without undue complexity.

As stated briefly above, a reason for the use of transmission lines, particularly of the *coaxial* variety, is that they can reduce or eliminate unwanted coupling between different parts of the circuit. When such coupling comes about from electric fields, those electric fields terminate on charges which remain on the outside of the outer conductor and do not become part of the voltage or currents which are flowing on the inside. When the unwanted coupling is via magnetic field, twin lines may be used in the *twisted pair* configuration so that the magnetic flux which links a part of the circuit is approximately cancelled by the magnetic flux linking another part, and only small net coupling exists.

Chapter 2

PROPERTIES OF TRANSMISSION LINES

2.1 Introduction

Having noted in the previous chapter the inadequacy of lumped circuit theory for the description of the properties of real electronic circuits, particularly at higher frequencies, we now proceed to the development of the first of the higher level theories that are of practical use. Thus this chapter is concerned with the development of the properties of transmission lines under the assumptions of what we call *distributed circuit theory.*

2.1.1 Types of structure

The signal propagation structures we consider can take the form of either *twin-line* or *coaxial cable*. An illustation of both of these transmission line stuctures is provided in Figure 2.1.

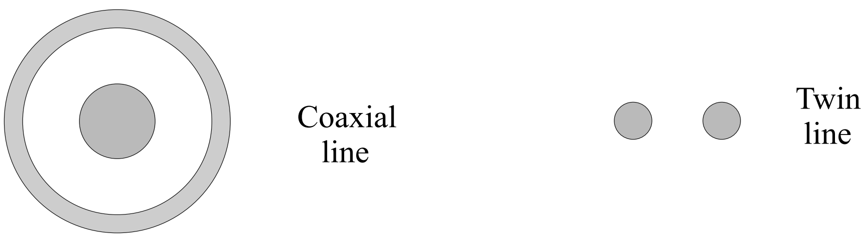


Figure 2.1: Coaxial line and twin line structures.

2.1.2 Assumptions

In our analysis we make a number of assumptions. The first of these is that the length of the line runs along a z axis and that the line characteristics are uniform with respect to the z axis. This second assumption is made to enable an appropriatly simple analysis.

The third assumption is that such magnetic fields as exist are confined to the *transverse plane*, that is the plane perpendicular to that z axis. This assumption enables us to conclude that the line integral of the electric field from a point on one of the conductors to another point in the same transverse plane on another conductor is independent of the path, and therefore one may speak meaningfully of the voltage between the two conductors in a plane defined by a particular value of the z co-ordinate.

The fourth assumption is that the currents which flow in the two conductors are equal in magnitude and opposite in direction.

We also assume that the currents flow uniformly on the *surfaces* of the conductors rather than being distributed over their cross section. Our reasons for making this assumption however can not be made clear until Chapter 9, dealing with the behaviour of electromagnetic fields in dissipative media, wherein the effect known as *skin effect* is introduced.

2.2 Basic Equations

2.2.1 Co-ordinate system

For our analysis we will use a coordinate system outlined in Figure 2.2

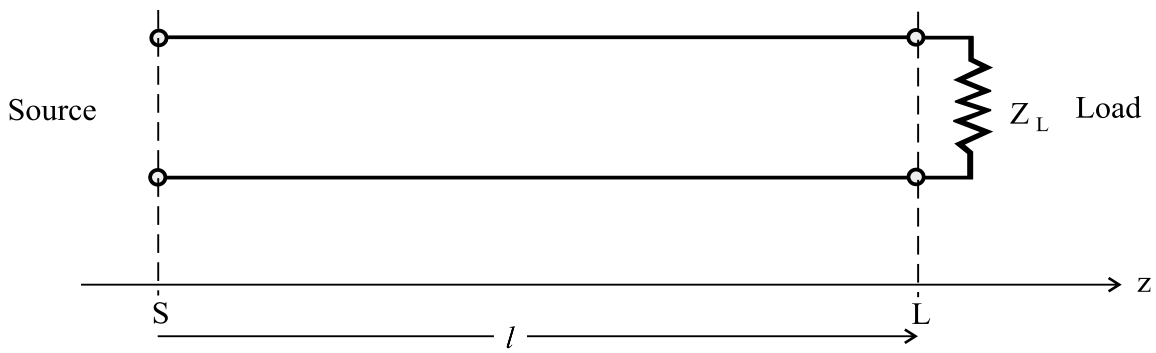


Figure 2.2: Co-ordinates and notation for transmission line analysis.

It is worth studying this diagram as it represents particular choices among many which are possible and the details of the equations to be developed depend upon these choices. We have followed in this diagram the majority but by no means universal tradition followed by writers of text books.

It may be noted in the diagram that the source is placed at the left of the load and that the z axis is directed from the source to the load. In respect of both these matters some text book writers make different choices, with the result that if the differences are not noted the fact that the resulting equations do not match those developed here will appear to be mysterious.

2.2.2 Differential equations for a short length

We assume the line is characterised by distributed resistance R, inductance L, capacitance C, and conductance G per unit length. Of these elements, the L and C provide energy storage, while the R and G provide energy dissipation.

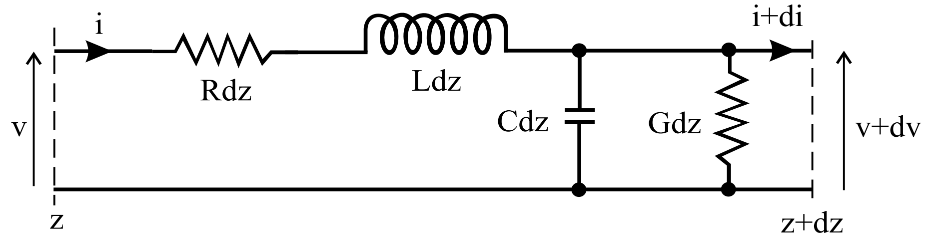


Figure 2.3: Voltages and currents in a short line length.

Applying Kirchhoff's voltage law to the small length dz of the line shown in Figure 2.3 gives the result

$$-dv = R\,dz\,i + L\,dz\,\frac{\partial i}{\partial t} \tag{2.1}$$

If we divide by dz and let dz tend to zero, and recognise that Figure 2.3 is an illustration of conditions on the line at one particular time t, the resulting derivative can be seen to be the partial derivative $\partial/\partial z$ with respect to position. This leads to the equation

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t} \tag{2.2}$$

Applying Kirchhoff's current law to the small length dz of the line gives the result

$$-di = G\,dz\,(v + dv) + C\,dz\,\frac{\partial (v + dv)}{\partial t} \tag{2.3}$$

If we divide by dz, and let dz tend to zero, and take note that dv will tend to zero as dz does, and again take note that Figure 2.3 is an expression of conditions on the line at one particular time t, the resulting derivative can again be seen to be a partial derivative with respect to position. This leads to the equation

$$\frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t} \tag{2.4}$$

These basic equations are difficult to solve for non-sinusoidal waves in the general case. Hence, we will study the particular cases of

- transients on lossless lines; and
- sinusoidal waves on lossy lines.

In each of these cases we obtain solutions of tolerable simplicity and significant utility.

2.3 Transients on Lossless Lines

2.3.1 Basic equations again

In a lossless line we put $R = 0$ and $G = 0$. Then equations 2.2 and 2.4 become

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} \tag{2.5}$$
$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} \tag{2.6}$$

2.3.2 Wave equation

If we eliminate i we obtain the one-dimensional wave equation

$$\boxed{\frac{\partial^2 v}{\partial z^2} = LC\frac{\partial^2 v}{\partial t^2}} \tag{2.7}$$

2.3.3 General solution

The general solution contains forward and reverse travelling waves of arbitrary shape, and has the form

$$\boxed{v(z,t) = V_f(z - ct) + V_r(z + ct)} \tag{2.8}$$

It may easily be confirmed that this solution satisfies equation 2.7 when the parameter c, which has the significance of the velocity of the wave in either the forward or backward direction, has the value

$$\boxed{c = \frac{1}{\sqrt{LC}}} \tag{2.9}$$

The general solution can be seen to be a linear combination of independent waves V_f and V_r travelling in the forward and backward directions resepctively. The velocities of the forward and backward waves are equal. The shapes of the forward and backward waves can in the general case be quite arbitrary and independent.

2.3.4 Illustration of solution

We chose for our first illustration the situation when only a forward wave is present. The wave may be illustrated against a time axis as shown in Figure 2.4. Because the function V_f is of the single argument $z - ct$, if the voltage observed at the origin of position as a function of time is represented by the solid curve in Figure 2.4, then at some position to the right of the origin the same voltage as a function of time will be observed but at a later time, as is shown by the dotted curve. As the observation point is moved progressively to the right to larger values of the position coordinate z, there will be an increase in the delay before the above defined voltage waveform is observed. All of these features are illustrated in Figure 2.4.

The situation on the line may alternatively be represented as shown in Figure 2.5, wherein at one particular time the voltage waveform for all points on the line is shown. This illustration is for a wave of the same shape as was illustrated in Figure 2.4. If one pictures the wave at later and later times, it will be seen to have moved further and further to the right.

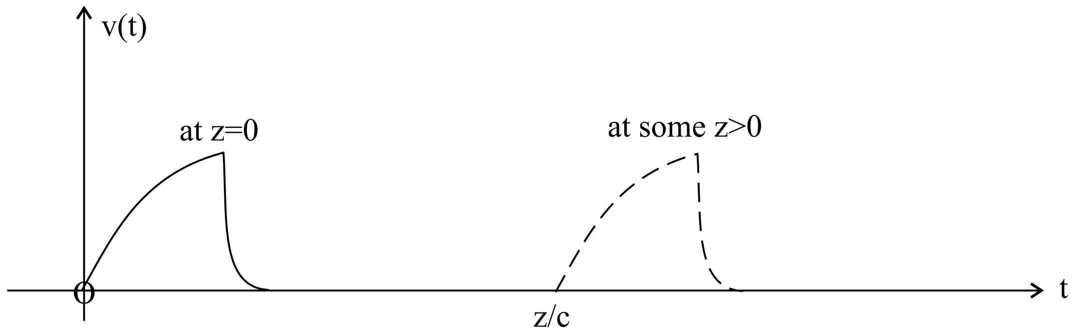


Figure 2.4: Illustration of a forward wave against a time axis.

It is interesting to note the shape of the wave in this latter diagram appears to be a mirror image of the one when drawn against the time axis. This is a natural consequence of the fact that the shape of the forward wave is an expressed as a function V_f of the single argument $z - ct$ and the coefficients of z and t in that combination have different signs.

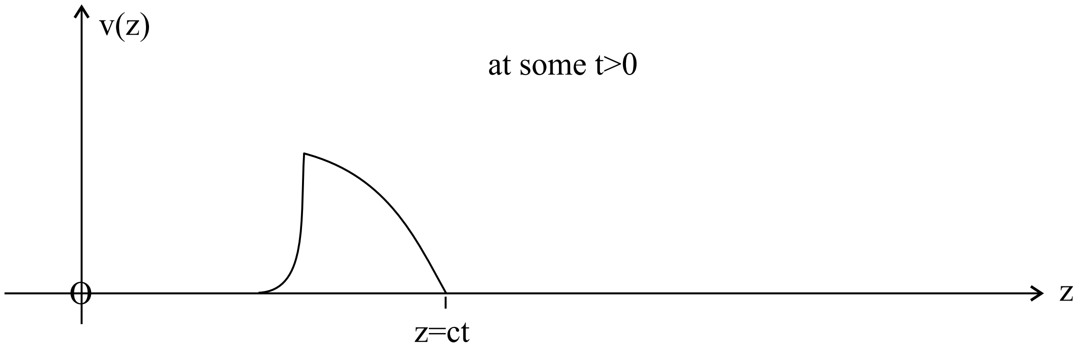


Figure 2.5: How the same wave looks in the position domain.

2.3.5 Exercise

A useful exercise for the reader would be the production of diagrams similar to those of Figures 2.4 and 2.5 illustrating against time and position axes respectively the reverse wave $V_r(z + ct)$.

2.3.6 Relation of voltage to current

So far we have pictured simply the voltage on the line, but that voltage is of course accompanied by a current. We now ask what relation this current has to the voltage.

To find the relation between voltage and current, we substitute v back in equation 2.6 and integrate with respect to time, and obtain

$$\sqrt{\frac{L}{C}}i(z,t) = V_f(z-ct) - V_r(z+ct) + f(z) \tag{2.10}$$

where f(z) is the constant of integration.

We substitute this result back into equation 2.6 and obtain the result $f'(z) = 0$, i.e. $f(z) =$ constant. This constant term corresponds to a steady d.c. voltage, which we set to zero, as we are not interested in a superimposed d.c. solution. Hence, the above relation becomes

$$Z_0 i(z,t) = V_f(z-ct) - V_r(z+ct) \tag{2.11}$$

where

$$\boxed{Z_0 = \sqrt{\frac{L}{C}}} \tag{2.12}$$

A little investigation will show that the parameter Z_0 has the *dimensions* of resistance and its *units* are thus ohms.

It is however common to call Z_0 the *characteristic impedance* rather than the characteristic resistance. This is done because because we will encounter, in a later section dealing with the analysis of sinusoidal waves on a transmission line, a concept which is properly called characteristic impedance and which has a defining equation which is formally identical with equation 2.12. It is traditional to recognise that homology by the use of the same terminology, even though doing so disguises the reality that in the present context Z_0 is really a *resistance*, and that in the present context the concept of *impedance* has no meaning.

Thus, the current can be written as the sum of forward and reverse components as

$$\boxed{i(z,t) = I_f(z-ct) + I_r(z+ct)} \tag{2.13}$$

where

$$\boxed{I_f = \frac{1}{Z_0}V_f} \tag{2.14}$$

$$\boxed{I_r = -\frac{1}{Z_0}V_r} \tag{2.15}$$

In the above equations, Z_0 is called the *characteristic impedance*, and its inverse $Y_0 = 1/Z_0$ is called the *characteristic admittance*.

Note that although V_f and V_r, and I_f and I_r are functions of $z-ct$ and $z+ct$ respectively, we often write them without arguments.

2.4 Reflections

In the previous section it was stated that the forward and backward waves may be entirely unrelated and of different shapes. This situation may be brought about if quite independent excitations are applied at both the source and the load of the line. More commonly, however, some excitation is provided at the source end and some kind of passive circuit elements are provided at the load. In this situation the forward and backward waves are not independent. The forward wave is provided by the excitation at the source end, while the reverse wave may best be thought of as having arisen from reflection of the forward wave at the load end. Investigation of this phenomenon is the subject of this section.

2.4.1 Co-ordinates and notation again

We consider the behaviour of a transmission line terminated as shown in Figure 2.6 in a *resistance* Z_L.

It might appear incongruous that we have chosen to represent what is a resistance by the symbol Z_L which is normally an indication of an impedance. The reason for this choice of notation is that we will encounter in a later section dealing with the analysis of transmission lines in the sinusoidal steady state, concepts and equations very similar to those being uncovered here. It is common to borrow from that different context the notation used therein although there is a certain inappropriateness in doing so.

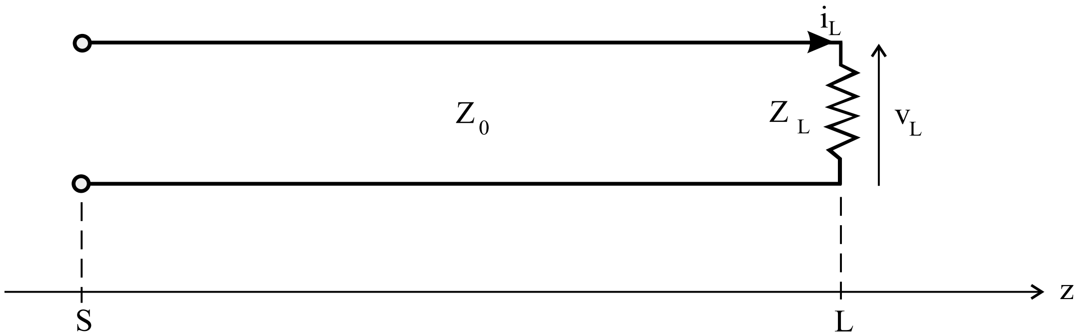


Figure 2.6: Transmission line terminated in a resistance Z_L.

Z_L could be a simple resistive termination, or could be, as we shall see later, another transmission line.

At the load $z = L$ we have

$$\begin{aligned} v &= V_f + V_r = v_L \\ \text{and} \qquad Z_0 i &= V_f - V_r = Z_0 i_L \end{aligned} \tag{2.16}$$

We re-arrange these equations, and obtain

$$\frac{V_f - V_r}{V_f + V_r} = \frac{Z_0}{Z_L} \tag{2.17}$$

We re-arrange to obtain, at the load

$$\frac{V_r}{V_f} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.18}$$

Thus, if V_f at the load is known, we can calculate V_r. Note that when $Z_L = Z_0$ there is no reflection. This occurs when the line is terminated in a resistance equal to its characteristic impedance.

Note that although we have not explicitly shown the arguments of V_f and V_r, V_f is a function of $z - ct$ and V_r is a function of $z + ct$. The equation just quoted has been obtained by examining conditions at the load, and thus applies for the particular position $z = L$, but for any time.

2.4.2 Voltage reflection factor

We define the *voltage reflection factor* of the load as

$$\Gamma_v(L) = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.19}$$

In the light of equation 2.18 above this becomes

$$\boxed{\Gamma_v(L) = \frac{V_r}{V_f}(\text{at load}) = \frac{Z_L - Z_0}{Z_L + Z_0}} \tag{2.20}$$

Note that despite our use of Z_0 and Z_L to denote the line and the load impedance, the defintion here only applies to *resistive* loads, because we have assumed $v_L = Z_L i_L$, and v_L and i_L are functions of time. However, we will see later that the definition 2.19 remains useful in the *sinusoidal steady state*, so we have been using Z in place of R for both Z_0 and Z_L.

2.4.3 Reflection factors for special cases

Note the values given in Table 2.1 for $\Gamma_v(L)$ obtained with various values of the load impdedance Z_L.

Condition	Z_L	$\Gamma_v(L)$
Matched	Z_0	0
o/c	∞	1
s/c	0	-1

Table 2.1: Reflection factors for special cases.

It is worth committing this table to memory. A good aid to doing so is to translate it into words such as: when the line is terminated in a resistance equal to its own characteristic impedance, the line is said to be *matched*, and there is no reflection, when the line is terminated in an open circuit, the reflection factor is one, and the reflected wave has the same amplitude and sign as the incident wave; and when the line is terminated in a short circuit, the reflection factor is minus one, and the reflected wave has the same amplitude as the incident wave, but has the opposite sign.

2.4.4 Uncharged lines

If we assume a line is initially uncharged, and we apply a signal at the source end, we will launch a forward wave, but we will have initially no backward wave, since there is no source at the load end, and the forward wave will not yet have reached the load to produce a reflection. Thus we assume that, at least for a time, $V_r = 0$ and $I_r = 0$ at the input.

The input resistance of the line when there is no reflected wave is then

$$R_{in} = \frac{v_{in}}{i_{in}} = \frac{V_f}{I_f} = \frac{Z_0 I_f}{I_f} = Z_0 \tag{2.21}$$

Thus, the input resistance of an initially uncharged line is initially equal to the characteristic impedance.

2.4.5 Very long lines

If the line were of infinite in length, this condition would be maintained forever.

2.5 Transients on Lossless Lines

We now begin our study of what happens when signals are first applied to a line or when the connections of impedances to a line are suddenly changed. This topic is known as the study of transients on lossless lines

2.5.1 Charging a finite length line through a resistor

In the first example we will consider what happens when an initially uncharged line is suddenly connected to an excitation voltage source through a resistance which it is common to call Z_S but which might have been more properly known as R_S. This situation is depicted in Figure 2.7

Because we have not considered any relation between the source resistance Z_S, the load resistance Z_L and the characteristic impedance Z_0, the situation we are studying has considerable generality. Its study will lead to the powerful concept of the *lattice diagram*, which is an orderly way of keeping track of multiple reflections up and down the line which occur after the excitation is first applied.

In the above circuit we close the switch $t = 0$ and let $T = l/c$ be the time for a one-way propagation down the line.

We expect that the closing of the switch will first launch, at the source end, a forward wave, which will travel over a time T to the load end, where it will be reflected, travel back over a time T to the source end, where it will be reflected forward again.

Consider first the sending end. At $z = S$ we have at all times

$$v_S = V_f(S,t) + V_r(S,t) \tag{2.22}$$

and

$$Z_0 i_S = V_f(S,t) - V_r(S,t) \tag{2.23}$$

But v_S and i_S must also satisfly the boundary conditions provided by the source, viz.

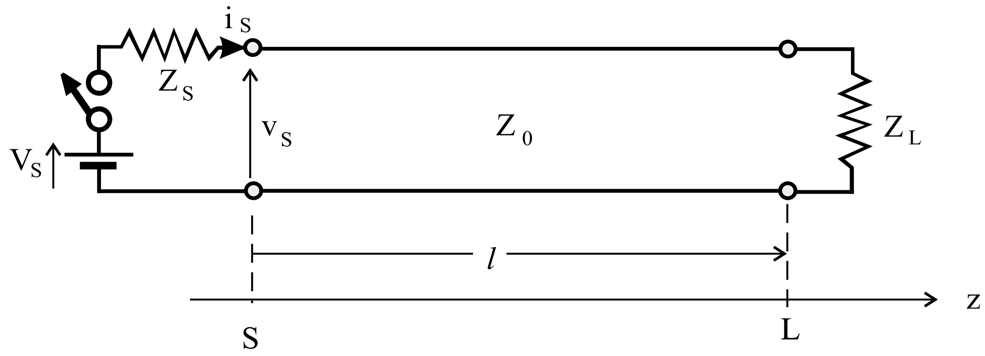


Figure 2.7: Circuit for charging a finite length line through a resistor.

$$v_S = V_S - Z_S i_S \tag{2.24}$$

.

If we eliminate v_S and i_S from the three equations above, we obtain

$$\boxed{V_f(S,t) = V_S \left(\frac{Z_0}{Z_S + Z_0}\right) + V_r(S,t) \left(\frac{Z_S - Z_0}{Z_S + Z_0}\right)} \tag{2.25}$$

This equation has a simple interpretation, using the idea of superposition. Omitting at first V_r, (and recognising that $V_r(S,t)$ will be zero for $0 \le t < 2T$) we have, as far as $V_f(S,T)$ is concerned, the equivalent circuit shown in Figure 2.8 to represent the first term of equation 2.25.

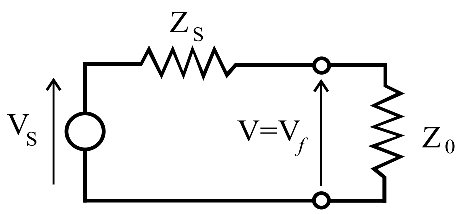


Figure 2.8: Equivalent circuit for launched wave.

Now the second term in equation 2.25 provides an extra contribution to $V_f(S,t)$ from $V_r(S,t)$ *multiplied by the reflection factor* $\Gamma_v(S)$ *looking back from the line to the source.*

Now consider the load end of the line. We have shown that $V_r(L,t) = V_f(L,t)\Gamma_v(L)$ where $\Gamma_v(L) = (Z_L - Z_0)/(Z_L + Z_0)$ is the reflection factor of the load. We may interpret all of these facts by means of the lattice diagram shown in Figure 2.9. The reader should take the time to study this figure in detail.

Adding all the terms which make a contribution to the load voltage we find the waveform at the load is

$$v_L(t) = V_S \left(\frac{Z_0}{Z_S + Z_0}\right) [1 + \Gamma_v(L)] \, [u(t-T) \quad + \quad \Gamma_v(L)\Gamma_v(S)u(t-3T)$$

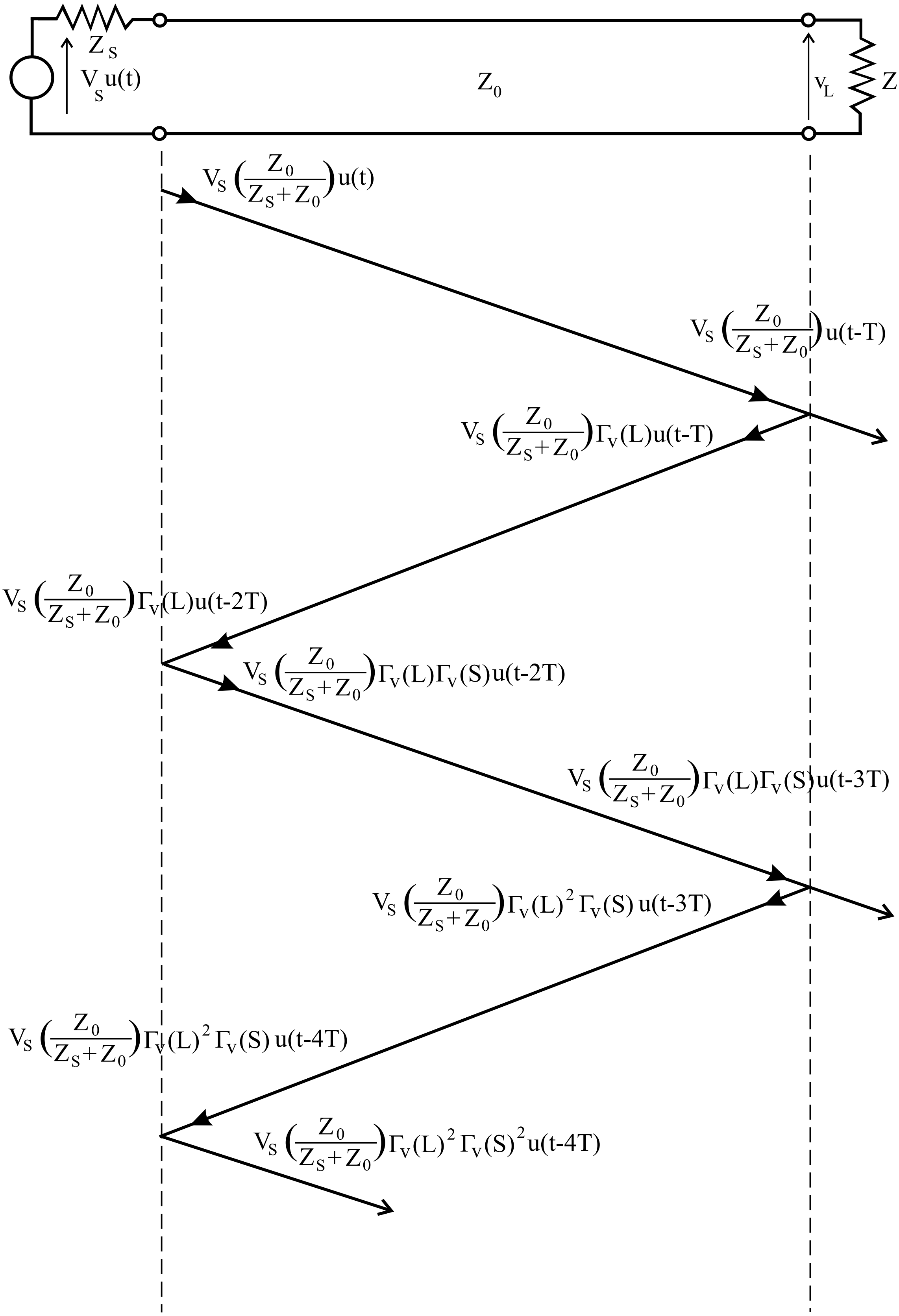


Figure 2.9: Lattice diagram.

$$+ \quad \Gamma_v(L)^2\Gamma_v(S)^2 u(t-5T) + ...\Big] \tag{2.26}$$

which we notice has the form of a *geometrical progression* with common ratio $\Gamma_v(S)\Gamma_v(L)$. For large values of t the final value of the load voltage may be shown by summing the geometric progression above to be

$$v_L(t) \to V_S \left(\frac{Z_0}{Z_S + Z_0}\right)[1 + \Gamma_v(L)]\frac{1}{1 - \Gamma_v(L)\Gamma_v(S)} \tag{2.27}$$

If we substitute for $\Gamma_v(L)$ and $\Gamma_v(S)$ in terms of Z_S, Z_L and Z_0 and re-arrange, we obtain the simple form

$$v_L(t) \to V_S \frac{Z_L}{Z_S + Z_L} \text{ as } t \to \infty \tag{2.28}$$

It is of course comforting that this value is exactly that which would be obtained if we had taken a simple view which indicates that if we wait long enough for all of the transient voltages present in this more detailed analysis to die away, we should obtain for the load voltage the same value as we would have obtained if we had used simple lumped circuit theory and regarded the transmission line as simply providing a connection between the source and the load and in which all of the travelling wave effects are ignored.

2.5.2 Line charging waveforms

The solution defined by equation 2.26 has the graph shown in Figure 2.10. The following observations may be made about the diagram.

- There is no activity at the load until the time T for a one-way propagation of signals on the line has elapsed.
- The initial step at that time is the product of two factors, one being the initially launched forward wave on the line and the other being the sum of unity and the reflection factor at the load, this sum being known as the *transmission factor* at the load junction.
- Although in this diagram it is shown that the initial step is to a voltage less than the eventual load voltage, the factors controlling the size of this initial step are such that in some circumstances the step is to a value greater than the eventual load voltage.
- Subsequent steps in the load voltage occur at times 2T, 4T, etc. after the initial step, that is at total times 3T, 5T, etc.
- Each of those subsequent steps is a common factor times the amplitude of the preceding step. That common factor is the product of the reflection factors looking back toward the load and looking towards the source.
- Those subsequent steps form a geometrical progression of which the common ratio happens to be positive in this illustration, but can be either positive or negative in the general case.

- The steps become progressively smaller so that the eventual load voltage converges towards a value which is recognisable as the value the load voltage would have if one simply regarded the source impedance and load impedance as forming a voltage divider delivering to the load a fraction of the source voltage. That is to say that the eventual load voltage is the value that would have been indicated if the transmission line effects had been ignored and simple lumped circuit theory had been used.

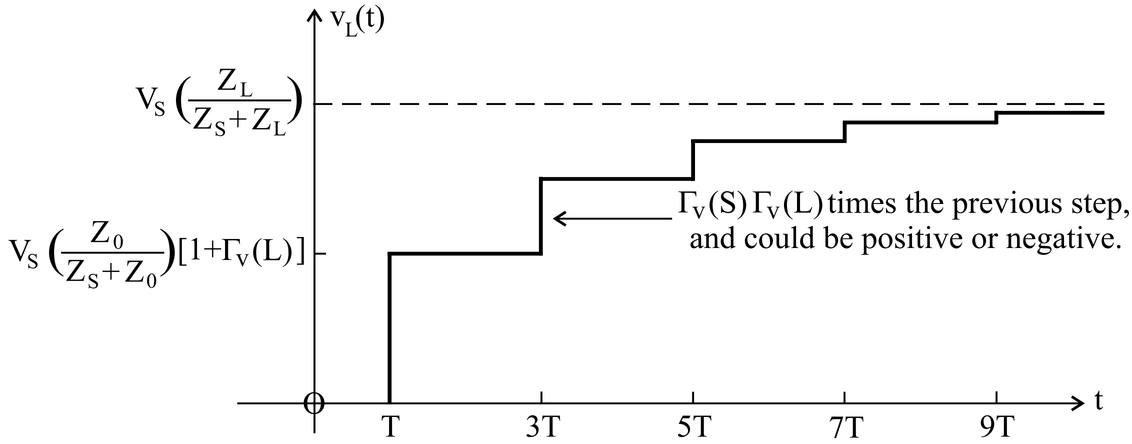


Figure 2.10: Line charging waveforms.

We discuss in the next section various *special cases* which can occur when the source and load impedances take particular values in relation to the line characteristic impedance.

2.5.3 Special cases

(a) Line matched at the load end

There are no reflections at all. The only wave present is a single forward wave. The total load voltage is as shown in Figure 2.11.

(b) Line matched at the source end

There are no reflections at the source. Hence, only the original forward wave and its single reflection are present. The total load voltage is as shown in Figure 2.12.

We note that $\Gamma_v(L)$ can be positive or negative. Cases of open circuit and short circuit terminations should be considered.

(c) Line open circuit at load end and fed from a low impedance source

There are many reflections from both the load and source ends of the line. The reflection factor at the load end is unity, and there is thus no attenuation of the wave reflected from the load end. The reflection factor at the source end is negative but only a little less than unity in magnitude. There is thus only small attenuation of the magnitude of the wave reflected at the source end. The total load voltage is as shown in Figure 2.13. There are many reflections before the final state is approached.

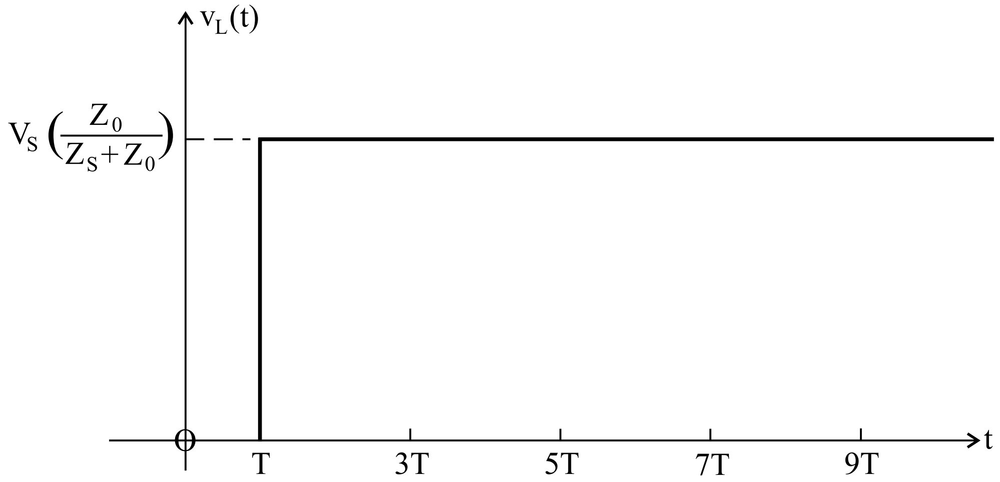


Figure 2.11: Waveform for a line matched at the load end.

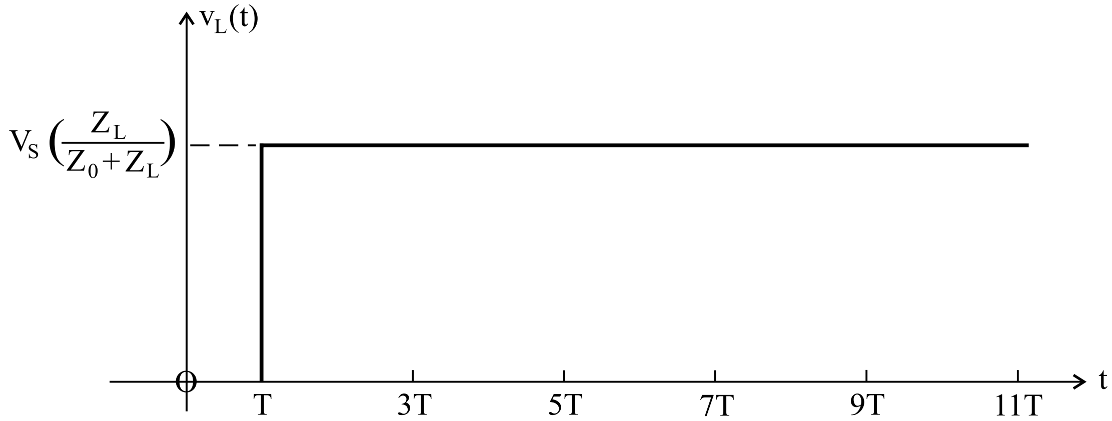


Figure 2.12: Waveform for a line matched at the source end.

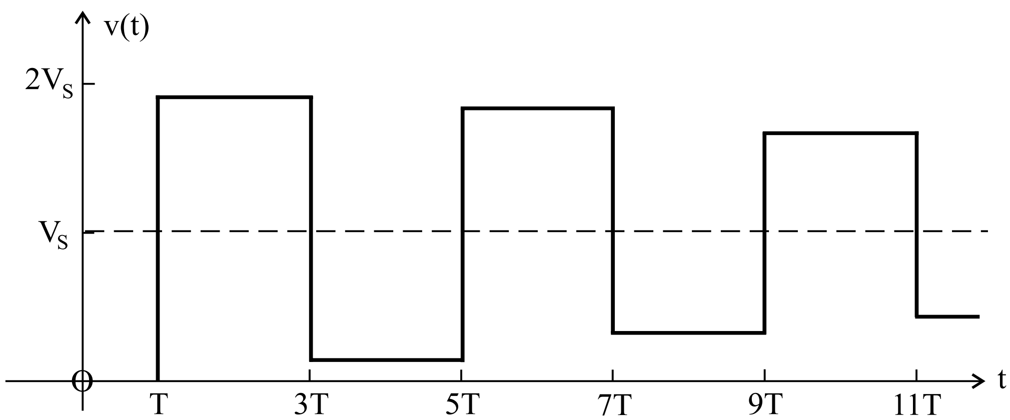


Figure 2.13: Waveform for an open circuit line.

2.5.4 Non-resistive terminations

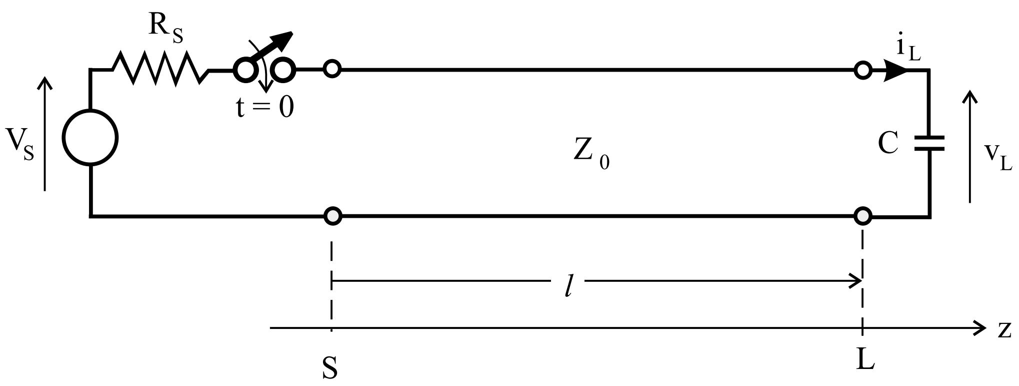


Figure 2.14: Line with non-resistive termination.

The situation becomes much more complex but still tractable when we consider non resistive terminations at the load end. As an *example* we will consider as shown in Figure 2.14 a transmission line terminated at the load end by a capacitance C. The structure is again excited from a voltage source V_S through a resistor R_S and a switch which is closed at time $t = 0$.

At the sending end we have no reverse wave for $0 \leq t < 2T$. Then we use the preiously described equivalent circuit

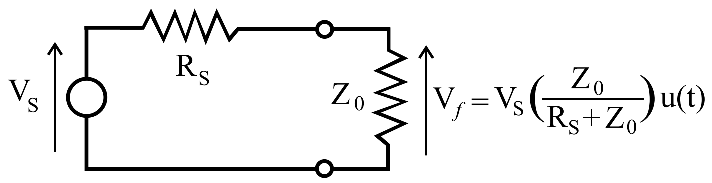


Figure 2.15: Equivalent circuit with no reverse wave.

to determine the initial forward wave. This wave reaches the load at $t = T$. However, with a non-resistive termination we cannot, in the time domain, define $\Gamma_v(L)$. Instead, we return to the basic equations at the load end, and write

$$v_L = V_f(L,t) + V_r(L,t) \tag{2.29}$$
$$Z_0 i_L = V_f(L,t) - V_r(L,t) \tag{2.30}$$

Adding these, we obtain

$$v_L + Z_0 i_L = 2V_f(L,t) \tag{2.31}$$

which has the equivalent circuit interpretation shown in Figure 2.16, where in the case for $0 \leq t < 2T$

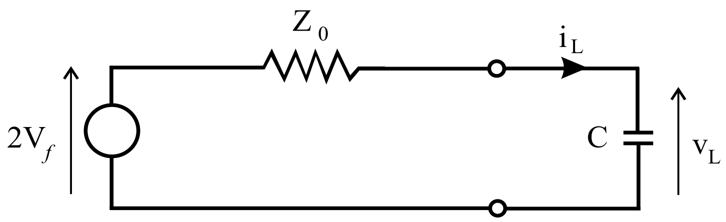


Figure 2.16: Equivalent circuit for the wave reaching the load

$$V_f = \frac{Z_0 V_S}{Z_0 + R_S} u(t - T). \tag{2.32}$$

This equivalent circuit indicates that the transmission line and its forward wave behaves *at the load end* in the same way as does the series combination of a voltage generator whose value is twice the value of the forward wave, in series with a resistor whose value is the characteristic impedance of the line. This equivalent circuit for a line feeding a load has applications wider than the current context, but it is that context which will continue to receive our attention.

To avoid excessive complexity, we will now consider the special case $R_S = Z_0$, for then there will be no reflecton at the source, and V_f given above will apply for all time.

It may be seen with aid of the above equation and equivalent circuit that the solution for v_L as a function of time is, with $\tau = Z_0 C$,

$$v_L = V_S u(t - T)(1 - e^{-(t-T)/\tau}) \qquad \text{for all } t. \tag{2.33}$$

The reverse wave produced at the load end can then be found from $V_r(L, t) = v_L(t) - V_f(L, t)$ and is

$$V_r(L, t) = V_S u(t - T)\left(\frac{1}{2} - e^{-(t-T)/\tau}\right) \qquad \text{for all } t.$$

To find the V_r at some point $z < L$ we add a further delay time $T - t'$ where $t' = z/c$ to obtain

$$V_r(z, t) = V_S u(t + t' - 2T)\left(\frac{1}{2} - e^{-(t+t^{\theta}-2T)/\tau}\right) \qquad \text{for all } t.$$

The total voltage on the line at any point and time is then obtained by adding to this backward wave the forward wave $V_f(z, t) = \frac{1}{2} V_S u(t - t')$ to obtain

$$v(z, t) = \frac{1}{2} V_S \left(u(t - t') + u(t + t' - 2T)(1 - 2e^{-(t+t^{\theta}-2T)/\tau})\right).$$

A summary of these results can be seen in diagram form in Figure 2.17. The different sections of this figure deserve study to confirm that they are in accord with our intuitive expectation for the behaviour of the overall system. In the upper part of the diagram we see for the case when $Z_S = Z_0$ a forward wave with a step front and amplitude of half the source voltage advancing towards the load.

When that voltage reaches the load, which is a capacitor and cannot sustain step changes in its voltage, there is reflected from the load, at least in the first instance, a step

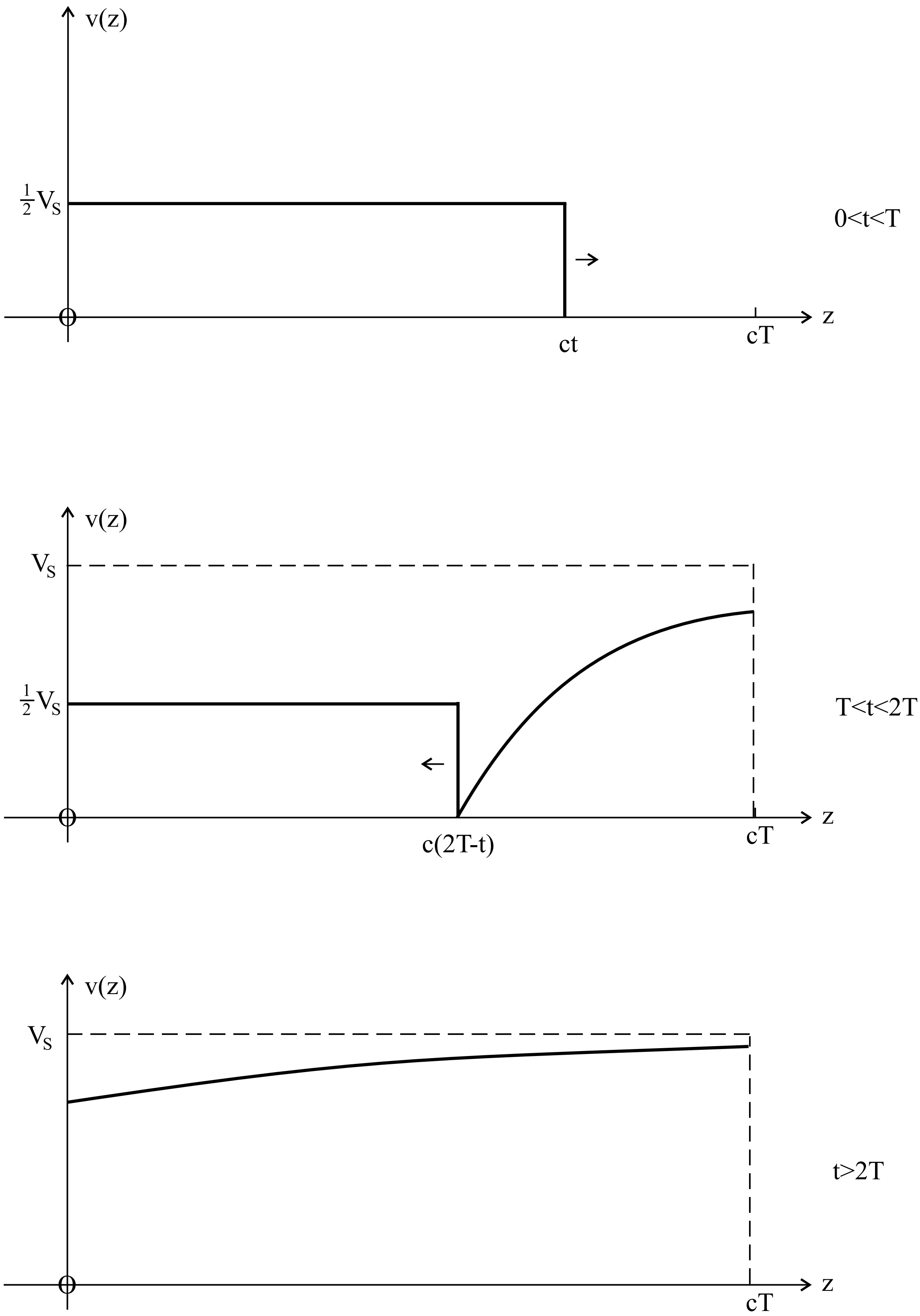


Figure 2.17: Summary of these results.

voltage of equal amplitude and opposite sign, which forms the initial reverse wave. The capacitor is seen as exhibiting its normal behaviour as far as transients go of behaving at least to instantaneous changes as a short circuit.

The capacitor does not, however, behave as a short circuit for all time. The capacitor will begin to charge, and develop a voltage which is seen at the right hand end of the centre diagram. The initial reflected wave of minus a half V_S will however continue to propagate towards the left where, in conjunction with the steady forward wave of one half V_S it will produce, at least at the point of meeting of these waves, a zero of voltage. To the left of that point, where the reflected wave has not yet reached, we will just have the forward wave of one half V_S. To the right of that point we will have the sum of that forward wave and a negative but steadily diminishing in amplitude reflected wave, as the capacitor behaves less and less like a short circuit. In fact as time evolves the capacitor's behaviour tends to that of an open circuit, for which the reflected wave is equal to the forward wave, and the total voltage tends to become the sum of two terms, one of which is half the source voltage and other of which is almost that, so that we see as in the right hand half of the middle diagram, and the bottom diagram, the capacitor voltage tending towards the source voltage.

This long term behaviour is again what we would expect on the basis of lumped circuit theory for a capacitance C being charged through a resistance Z_0.

2.5.5 Exercise

Derive a corresponding solution for a line terminated by an *inductance*.

2.6 Analysis in the Frequency Domain

At the end of Section 2.3 we noted that the general equations 2.2 and 2.4 which describe waves on transmission lines can be usefully solved in two particular contexts, one being transmission lines containing neither of the energy loss producing elements R or G, and the other being transmission lines which do contain the full set of elements R, L, C, and G but in which the time waveform is restricted to the sinusoidal steady state. It is to that particular context to which we turn now in our study.

Thus we now return to the general case of lossy lines, and persue an analysis in the *sinusoidal steady state*.

2.6.1 Phasor Notation

We remind ourelves that the relation between time domain and frequency domain (phasor) variables is expressed by

$$v(z,t) = \Re e\left\{\mathrm{V}(z)e^{j\omega t}\right\} \tag{2.34}$$

V(z) is a *complex phasor* representing *peak* value, not r.m.s. It does not have a time variation. Some authors introduce the additional quantity

$$\hat{\mathrm{V}}(z) = \mathrm{V}(z)e^{j\omega t} \tag{2.35}$$

and in so doing introduce unnecessary confusion by labelling that quantity as a phasor. We decry this behaviour and we will not imitate it in these notes. In these notes phasors are represented by capital Roman type such as V, and as frequently stated have no time variation.

2.6.2 Transformation of equations

Transforming equations 2.2 and 2.4 to the frequency domain for the sinusoidal steady state gives

$$\frac{d\mathrm{V}}{dz} = -(R + j\omega L)\mathrm{I} = -Z\mathrm{I} \tag{2.36}$$

$$\frac{d\mathrm{I}}{dz} = -(G + j\omega C)\mathrm{V} = -Y\mathrm{V} \tag{2.37}$$

where we have introduced the *series impedance* and *shunt admittance* each per unit length

$$Z = \quad R + jX \quad = R + j\omega L \tag{2.38}$$

$$Y = \quad G + jB \quad = G + j\omega C \tag{2.39}$$

2.6.3 Solution of the equations

If we eliminate I between equations 2.36 and 2.37 we obtain

$$\frac{d^2\mathrm{V}}{dz^2} = \gamma^2\mathrm{V} \tag{2.40}$$

where

$$\boxed{\gamma = \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)}} \tag{2.41}$$

is called the *complex propagation constant.*

We note that the equation above gives two values for γ, of which one must be in the first quadrant, and the other in the third. We denote the value in the first quadrant by γ, and its real and imaginary parts by α and β. The value in the third quadrant is called $-\gamma$.

A solution of the equation 2.41 is

$$\boxed{\mathrm{V}(z) = \mathrm{V}_f e^{-\gamma z} + \mathrm{V}_r e^{+\gamma z}} \tag{2.42}$$

It is always useful to observe similarities and differences between various elements of notation. The variables V_f and V_r which have been just introduced may be contrasted with the time domain variables V_f and V_r in that the frequency domain variables just introduced are *complex constants* with no spatial or time variation whereas the time domain variables V_f and V_r are not constants but are arbitrary functions of the single implied argument $z - ct$ or $z + ct$. Moving on, we note that the subscripts f and r are chosen to reflect the fact that, as we will show later, V_f represents the amplitude and

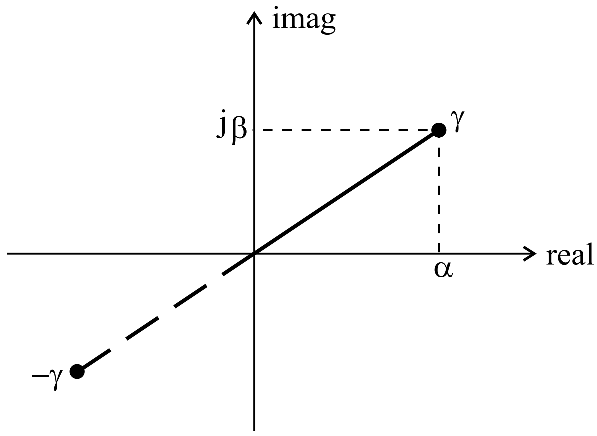


Figure 2.18: Argand Diagram for γ.

the phase (at the origin) of a forward wave, while V_r represents the amplitude and phase (both at the origin) of a reverse wave. As already noted we call $\gamma = \alpha + j\beta$ the complex propagation constant. We call α the *attentuation constant* and β the *phase constant*. The current $\mathrm{I}(z)$ which accompanies this voltage can be found from equation 2.36 to be

$$\mathrm{I}(z) = -\frac{1}{Z}\left[-\gamma \mathrm{V}_f e^{-\gamma z} + \gamma \mathrm{V}_r e^{+\gamma z}\right] \tag{2.43}$$

We substitute for γ from 2.41, and obtain

$$\begin{aligned} \mathrm{I}(z) &= \sqrt{\frac{Y}{Z}}\left[\mathrm{V}_f e^{-\gamma z} - \mathrm{V}_r e^{+\gamma z}\right] \\ &= \frac{1}{Z_0}\left[\mathrm{V}_f e^{-\gamma z} - \mathrm{V}_r e^{+\gamma z}\right] \end{aligned} \tag{2.44}$$

where we have introduced the definition

$$\boxed{Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}} \tag{2.45}$$

which we call the *characteristic impedance* of the line. Note that Z_0 is complex, but if we examine carefully the derivation above we see that it first emerged as a notation for the quantity Z/γ. Since both Z and γ are in the first quadrant of the Argand diagram, Z_0 must be in the *right half plane*. Thus, when we derive Z_0 from equation 2.45, where we appear to have two solutions at our disposal, we must take the one *with a positive real part* to be consistent with our choice of γ as the value in the first quadrant.

2.6.4 Interpretation of the solution

We consider in turn each of the terms of the solution 2.44 above, and plot the real voltage for $t = 0$, and for some small time δt later. The results appear in Figures 2.20 and 2.21 respectively.

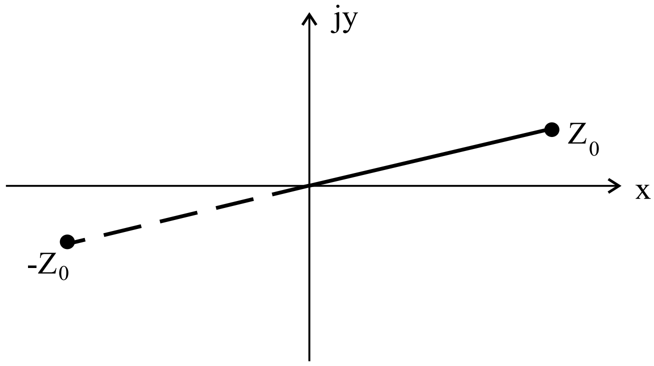


Figure 2.19: Argand diagram for Z_0.

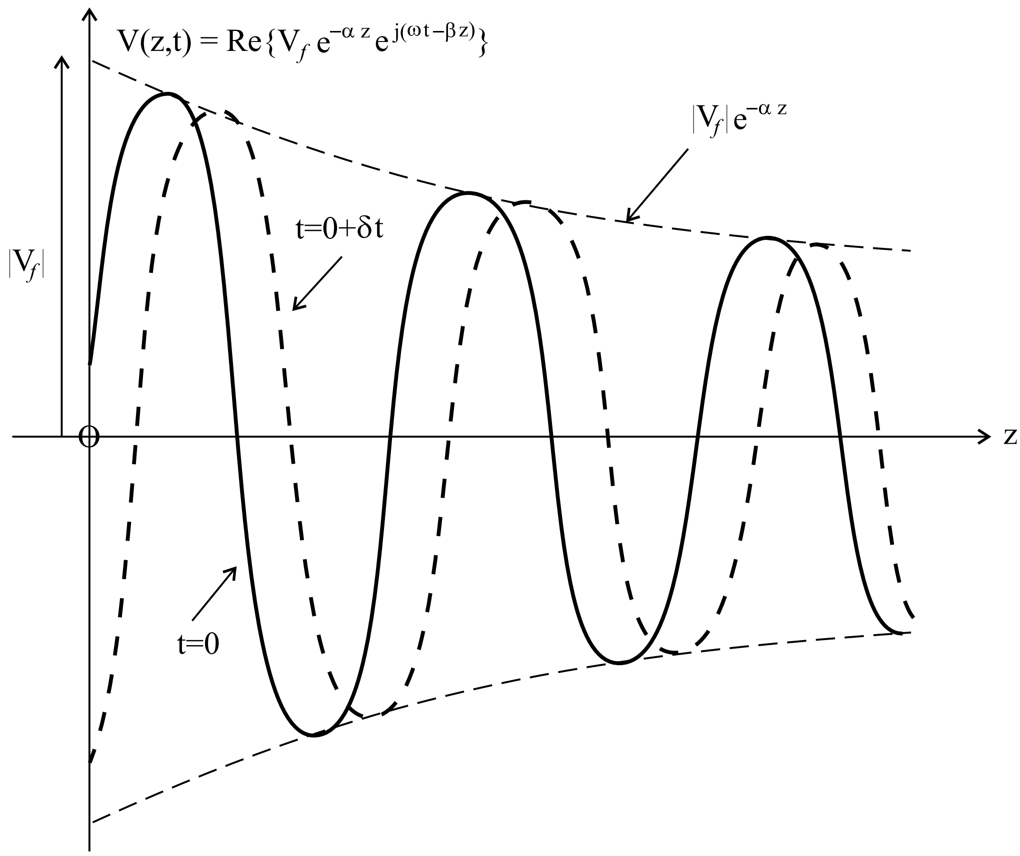


Figure 2.20: Illustration of a forward wave on a line.

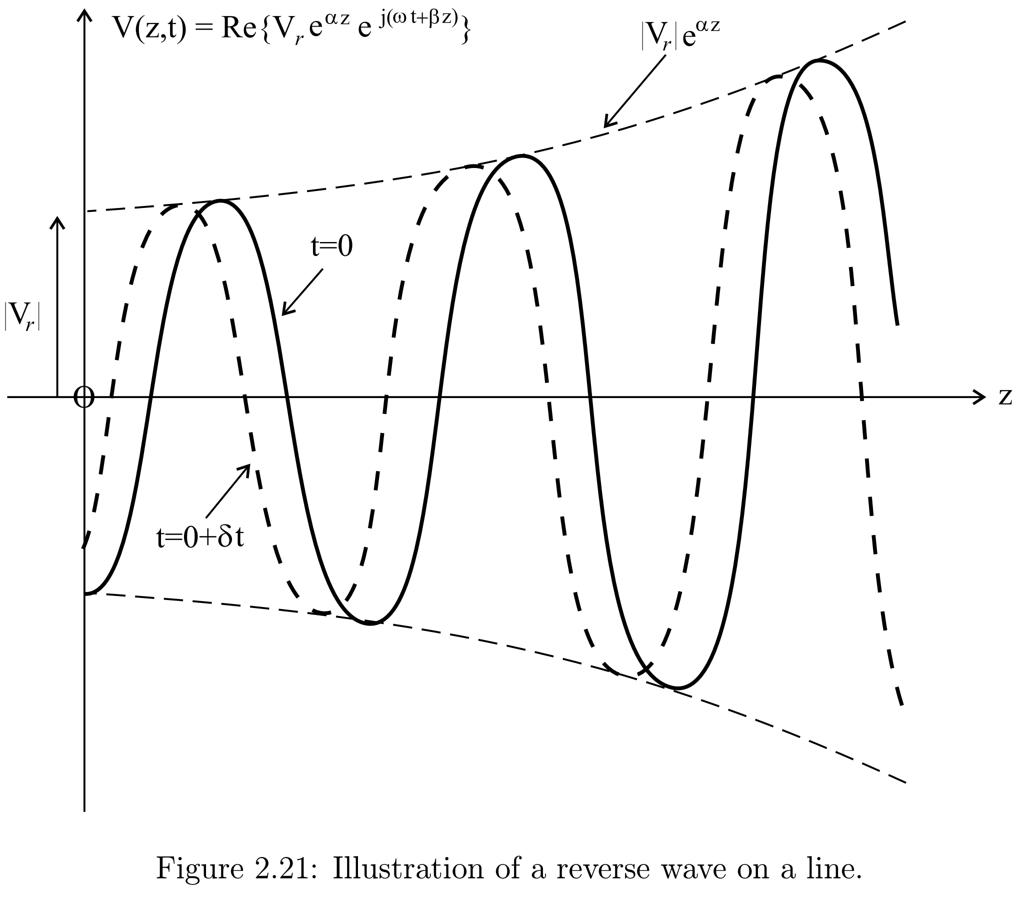


Figure 2.21: Illustration of a reverse wave on a line.

In each of thee diagrams, the wavelength is equal to the distance between zeros and is $2\pi/\beta$.

2.6.5 Summary of solution

Thus, we may summarise the general solution of the equations 2.36 and 2.37 as

$$\boxed{\mathrm{V}(z) = \mathrm{V}_f e^{-\gamma z} + \mathrm{V}_r e^{+\gamma z}} \tag{2.46}$$

$$\boxed{\mathrm{I}(z) = \mathrm{I}_f e^{-\gamma z} + \mathrm{I}_r e^{+\gamma z}} \tag{2.47}$$

$$\boxed{\mathrm{I}_f = \frac{\mathrm{V}_f}{Z_0}} \tag{2.48}$$

$$\boxed{\mathrm{I}_r = \frac{-\mathrm{V}_r}{Z_0}} \tag{2.49}$$

If we wish to express both the voltage and current in terms of the travelling waves of voltage alone we may write

$$\boxed{\mathrm{V}(z) = \mathrm{V}_f e^{-\gamma z} + \mathrm{V}_r e^{+\gamma z}} \tag{2.50}$$

$$\boxed{\mathrm{I}(z) = \frac{\mathrm{V}_f e^{-\gamma z} - \mathrm{V}_r e^{+\gamma z}}{Z_0}} \tag{2.51}$$

Evaluation of V_f and V_r (or I_f and I_r) must proceed from *boundary conditions*. This matter is covered in the next section.

2.7 Voltage Reflection Factor

In the time domain analysis we encountered the concept of voltage reflection factor. In that context it was a real constant. In the frequency domain analysis we will encounter a corresponding concept which we will give the same name, but in the frequency domain analysis it becomes a complex number. The use of the same name for closely related but not identical concepts is a tradition which is well established and does no harm to the reader who understands its origin. It sits well with the fact that formulae involving voltage reflection factor in the two different contexts are of the same structure. It is, however, still a violation of strict logic.

2.7.1 Boundary Conditions

The complete solution is known when V_f and V_r are known. We need two equations to find them; these can come from any two of the below quantities

1. Sending end voltage.
2. Sending end current.

3. Receiving end voltage.
4. Receiving end current.
5. Receiving end impedance.

Normally the receiving end impedance and one of the others is available.

We find it useful to extend our knowledge of impedance to other points on the line. To do this we find it convenient to first define the concept of *voltage reflection factor*. The definition introduced is a generalisation of the quantity already defined in the time domain for resistive terminations. We will find that voltage reflection factor is one of the most important concepts in transmission line theory, and it will be seen to supplant a purely impedance viewpoint.

2.7.2 Definition

We define the complex voltage reflection factor $\Gamma_v(z)$ at any point on the line as

$$\boxed{\Gamma_v(z) = \frac{\text{complex amplitude of the reverse voltage wave at } z}{\text{complex amplitude of the forward voltage wave at } z}} \tag{2.52}$$

This concept is important in that it is, as we shall shortly see, a lot simpler to visualise than solutions for total voltage current or impedance.

2.7.3 Variation of $\Gamma_\mathsf{v}(z)$ with position

Taking into account the variation of forward and backward waves with position we have

$$\boxed{\Gamma_v(z) = \frac{\mathrm{V}_r e^{\gamma z}}{\mathrm{V}_f e^{-\gamma z}} = \Gamma_v(0)e^{2\gamma z}} \tag{2.53}$$

When $z = L$, i.e. at the load, we denote Γ_v by $\Gamma_v(L)$, the reflection factor of the load. When $z = S$, i.e. at the source, we denote Γ_v by $\Gamma_v(S)$, the reflection factor *looking into the line* at the source end. From equation 2.53, then

$$\boxed{\frac{\Gamma_v(S)}{\Gamma_v(L)} = \frac{e^{2\gamma S}}{e^{2\gamma L}} = e^{-2\gamma(L-S)} = e^{-2\gamma l}} \tag{2.54}$$

It may be worth emphasising that in equation 2.53 the exponent on the right hand side is positive, whereas in equation 2.54 the exponent on the right hand side is negative. Both of these signs correctly express the fact that the voltage reflection factor becomes *retarded in phase* as we move back from the load.

It will hopefully be explained in lectures that this behaviour is as expected, and can be used as an aid to memory of the sign of the exponents in equation 2.53 or equation 2.54.

2.7.4 Impedance at any point

We define the *impedance at any point* by

$$Z(z) = \frac{\mathrm{V}(z)}{\mathrm{I}(z)} = \frac{\mathrm{V}_f e^{-\gamma z} + \mathrm{V}_r e^{+\gamma z}}{\mathrm{I}_f e^{-\gamma z} + \mathrm{I}_r e^{+\gamma z}} \tag{2.55}$$

This is the impedance we see if we cut the line at z and look to the right, as shown in Figure 2.22.

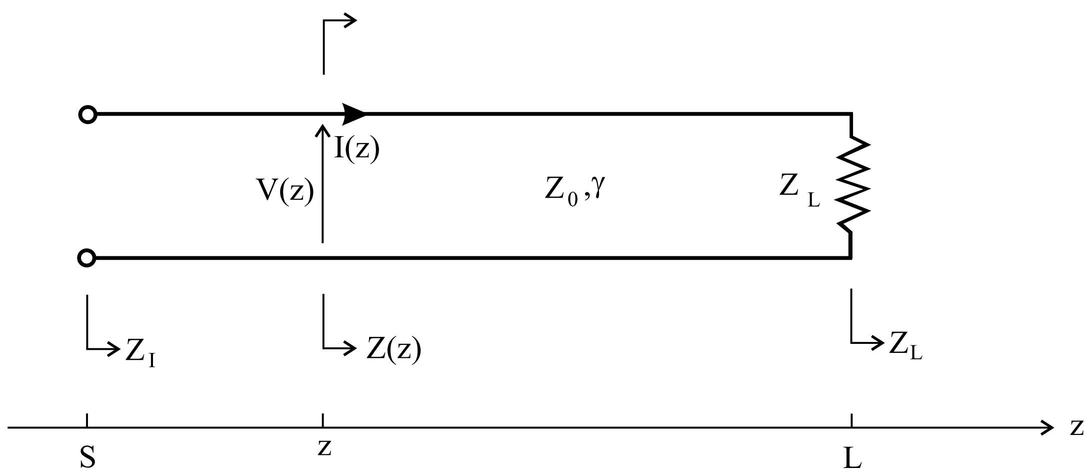


Figure 2.22: Impedance of the line, i.e. looking to the right.

We now divide the numerator and denominator of the right hand side of equation 2.55 by $\mathrm{V}_f e^{-\gamma z}$, and divide both sides by Z_0, and obtain

$$\boxed{\frac{Z(z)}{Z_0} = \frac{1+\Gamma_v(z)}{1-\Gamma_v(z)}} \tag{2.56}$$

This relation may be inverted to obtain

$$\boxed{\Gamma_v(z) = \frac{Z(z)-Z_0}{Z(z)+Z_0}} \tag{2.57}$$

In the special case when $z = L$

$$\boxed{\Gamma_v(L) = \frac{Z_L - Z_0}{Z_L + Z_0}} \tag{2.58}$$

Hence, given Z_L, to find the input impedance of the line we follow the steps

1. Calculate $\Gamma_v(L)$ from equation 2.58.
2. Calculate $\Gamma_v(S)$ from equation 2.54.
3. Calculate Z_I from equation 2.56.

If we combine all these steps algebraically, we find the result for Z_I is

$$\boxed{\frac{Z_I}{Z_0} = \frac{1 + \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-2\gamma l}}{1 - \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-2\gamma l}}} \tag{2.59}$$

The right hand side of this result simplifies to give

$$\boxed{\frac{Z_I}{Z_0} = \frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l}} \tag{2.60}$$

This is a very general result; it applies to a line of arbitrary length, losses and load impedance.

2.7.5 Matching

If a line is terminated in its characteristic impedance (which is complex for an arbitrary lossy line), i.e. if $Z_L = Z_0$, then equation 2.60 shows that

$$Z_I = Z_0 \qquad \text{for any } l$$

i.e. the input impedance becomes *independent of line length*. This is an important practical property, and the line is said to be *matched*.

2.8 Lossless Transmission Lines

We continue in this section with our analysis in the frequency domain, but are now prepared to make the further assumption that the line has no losses.

2.8.1 General results

We assume $R = 0$ and $G = 0$, and review previous results. We find that

$$\begin{aligned} \alpha &= 0 && \text{i.e. no attenuation} \\ \beta &= \omega\sqrt{LC} && \text{i.e no dispersion} \\ \text{as} \qquad v_p &= \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \text{constant} \\ v_g &= \frac{\partial\omega}{\partial\beta} = \frac{1}{\sqrt{LC}} = \text{same constant} \end{aligned} \tag{2.61}$$

The concepts of phase and group velocity appearing in the equations above should be reviewed in lectures, and students should ask that such review occur.

Further, we find that Z_0 is now *real*, and *independent of frequency*; in particular

$$\boxed{Z_0 = \sqrt{\frac{L}{C}}} \tag{2.62}$$

We find that the voltage reflection factor

$$\boxed{\Gamma_v(z) = \Gamma_v(0)e^{2j\beta z}} \tag{2.63}$$

changes in phase but not in magnitude as we go along the line. It *advances* in phase as we move along the $+z$ direction *toward* the load. At the source

$$\boxed{\Gamma_v(S) = \Gamma_v(L)e^{-2j\beta l}} \tag{2.64}$$

distant l back from the load the voltage reflection factor is *retarded* in phase as we make the line longer and move *back* from the load.

Note that the magnitude does not change, a fact in accordance with expected energy conservation along the lossless line. The general input impedance expression becomes

$$\boxed{\frac{Z_I}{Z_0} = \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l}} \tag{2.65}$$

2.8.2 Special cases

The input impedance of a transmission line for various special cases of termination at the load end is shown in Table 2.2

Case	Impedance
Short circuit load	$Z_I = jZ_0 \tan\beta l$
Shorted $\lambda/4$ Line	$Z_I \to \infty$ i.e. o/c
Open circuit load	$Z_I = -jZ_0 \cot\beta l$
Open Circuit $\lambda/4$ Line	$Z_I = 0$ i.e. s/c

Table 2.2: Impedance of various lossless transmission lines.

The results in the Table 2.2 are worthy of being committed to memory. They indicate that lines of various lengths are useful as *variable reactances*. When they are so used they are known as *stubs*. Stubs can be used for *tuning*, say at the end of a line.

2.8.3 Quarter wave lines

When $l = \lambda/4$, $\beta l = \pi/2$. Then

$$Z_I = \frac{Z_0^2}{Z_L} \tag{2.66}$$

This very important result means $\lambda/4$ lines can be used as *transformers*.

2.8.4 Example

An illustration of the use of the transmission line of quarter wave transformer is given in Figure 2.23. Here it is desired to transform a load resistance of 50 ohm to the value of 100 ohm which would provide the optimum load impedance for extracting power from the voltage source with an internal impedance of 100 ohm.

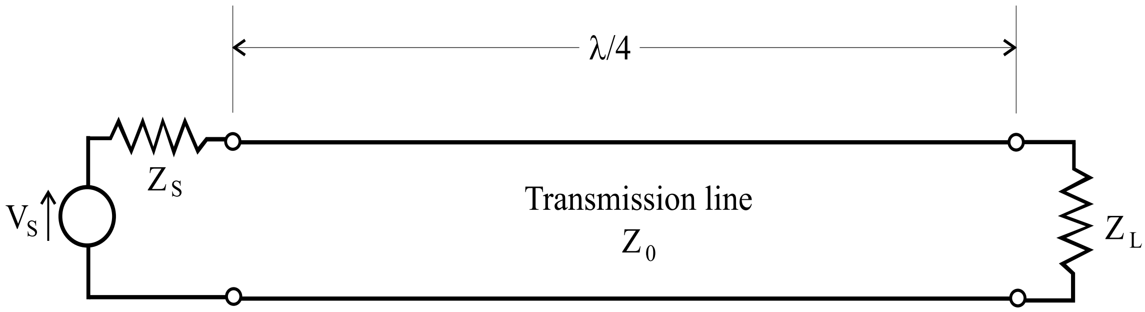


Figure 2.23: Matching using a quarter wave transformer.

To perform this transformation a transmission line, whose length is a quarter of a wave length at the operating frequency, is inserted between the load and the source. Using equation 2.66 we can see that the appropriate value of characteristic impedance is given by

$$\boxed{Z_0 = \sqrt{Z_I Z_L}} \tag{2.67}$$

For the values present in this problem, the appropriate line characteristic impedance is found to be 70.7 ohm.

2.8.5 Normalised impedance

We define the *normalised impedance* z corresponding to an actual impedance Z as

$$\boxed{z = \frac{Z}{Z_0}} \tag{2.68}$$

This normalised z is *dimensionless*. It is convenient parameter to use because Γ_v is also dimensionless. The relations between Γ_v and z are found to be

$$\boxed{\Gamma_v = \frac{z-1}{z+1}} \tag{2.69}$$

and

$$\boxed{z = \frac{1+\Gamma_v}{1-\Gamma_v}} \tag{2.70}$$

2.9 Admittance Formulation

While some circuit theory problems are easily solved in terms of impedances, others are easily solved in terms of admittances. This difference comes about because the rules for combining impedances in series are simple, but for combining impedances in parallel are complicated. The rules for combining admittances in parallel are simple, whereas the rules for combining them when they are in series are complicated. Whichever is the best formlation to use depends very much upon whether there is a preponderance of series or parallel connections in the situation being analysed. It is therefore appropriate that as well as exploring the impedance formlation as we have just done, we also in the present section explore the corresponding admittance formulation for the solution of transmission line problems.

We remark that in developing this formlation we are maintaining the assumption that the lines contain no loss.

2.9.1 General formula

For every impedance Z we have a corresponding admittance $Y = 1/Z$. It is easy to show, and should be taken as an *exercise*, that in admittance terms the formula 2.65 for input conditions becomes

$$\boxed{\frac{Y_I}{Y_0} = \frac{Y_L \cos\beta l + jY_0 \sin\beta l}{Y_0 \cos\beta l + jY_L \sin\beta l}} \tag{2.71}$$

This turns out to have the same form as equation 2.65 with all Z changed into Y.

2.9.2 Special cases (again)

The input admittance of a transmission line for various special cases of termination at the load end is shown in Table 2.3

Case	Admittance
Open circuit load	$Y_I = jY_0 \tan\beta l$
Open Circuit $\lambda/4$ Line	$Y_I \to \infty$ i.e. s/c
Short circuit load	$Y_I = -jY_0 \cot\beta l$
Shorted $\lambda/4$ Line	$Y_I = 0$ i.e. o/c

Table 2.3: Admittances of various transmission lines.

The values in this table are worthy of being committed to memory along with the values in Table 2.2. It is also worth comparing the tables to confirm that the second is a restatement of the same facts as in the first, but in admittance terms.

2.9.3 Quarter wave lines

For lossless transmission lines of length equal to a quarter of a wave length the operating frequency, the relation between load and input admittances is

$$\boxed{Y_I = \frac{Y_0^2}{Y_L}}. \tag{2.72}$$

2.9.4 Normalised admittance

We define the normalised admittance y as

$$\boxed{y = \frac{Y}{Y_0}}. \tag{2.73}$$

We find that $y = 1/z$. The relations between y and Γ_v are found to be of the form

$$\boxed{-\Gamma_v = \frac{y-1}{y+1}} \tag{2.74}$$

and

$$\boxed{y = \frac{1-\Gamma_v}{1+\Gamma_v}}. \tag{2.75}$$

These do not quite correspond in form to the impedance relations 2.69 and 2.70. We obtain more closely corresponding relations by defining below a *current reflection factor*, which is the mathematical dual of the previously defined voltage reflection factor.

2.9.5 Current reflection factor

The current reflection factor Γ_i is defined by

$$\boxed{\Gamma_i = \frac{\mathrm{I}_r e^{\gamma z}}{\mathrm{I}_f e^{-\gamma z}}}. \tag{2.76}$$

This definition is the obvious dual of the previously given definition in equation 2.52 for voltage reflection factor. Substituting for I_f and I_r in terms of V_f and V_r from equations 2.48 and 2.49 we find that

$$\boxed{\Gamma_i = -\Gamma_v}. \tag{2.77}$$

Then the relations between Γ_i and y are seen to be

$$\boxed{\Gamma_i = \frac{y-1}{y+1}} \tag{2.78}$$

$$\boxed{\text{and} \qquad y = \frac{1+\Gamma_i}{1-\Gamma_i}}. \tag{2.79}$$

These results now correspond in form to those of equations 2.70 and 2.69, a fact we will make use of when we study in Chapter 3 the Smith Chart.

Note that the current reflection factor Γ_i transforms along the line in the same way as does the voltage reflection factor Γ_v, i.e.

$$\boxed{\Gamma_i(z) = \Gamma_i(0)e^{2j\beta z}} \tag{2.80}$$

or

$$\boxed{\Gamma_i(S) = \Gamma_i(L)e^{-2j\beta l}}. \tag{2.81}$$

We again note the different signs of the exponents in these two equations, and the fact that each of them expresses the fact that the phase of the reflection factor is *retarded* as we move back from the load.

2.10 Voltage Standing Wave Ratio

We now begin the study of a parameter, namely the *voltage standing wave ratio*, of considerable theoretical and practical importance in transmission line theory. Its theoretical importance lies in the fact that it sharply illustrates the complexity of the variation and total voltage and current along the transmission line. Its practical importance lies in the fact that it is in fact a parameter which is most amenable to accurate experimental measurement in determining the impedance conditions which apply on a line.

It may be worth explaining here just why it is that measurements of total voltage and current in distributed systems is not easy. This difficulty is related to the fact that, as we have already learned, interconnecting wires have both distributed inductance and capacitance, and the very connections between points on a transmission line and a test instrument complicate the determination of what is happening on the line in two ways.

Firstly there is the usual phenomenon that a test instrument places a load across the circuit being measured, and in so doing disturbs the condition on the circuit so that we are no longer determining what we had wished to determine, which is the conditions on the undisturbed circuit. Secondly, the disturbance to the circuit is not easily predictable, because even if we know the load presented by the measuring instrument at its input terminals, the load becomes transformed by the interconnecting wires to a different value at the point where they are applied to the circuit. Thirdly, the voltage at the connection point becomes transformed along the interconnecting wires so the voltage at the input terminals to the measuring instrument is different from that applying on the line.

Fortunately the normal method of measurement of voltage standing ratio is free of all of these difficulties.

2.10.1 Voltage variation along a line

We look at the way the total voltage V(z) varies along the lossless line. We have from equation 2.42, with $\gamma = j\beta$

$$\mathrm{V}(z) = \mathrm{V}_f e^{-j\beta z} + \mathrm{V}_r e^{+j\beta z} \tag{2.82}$$

where V_f and V_r are complex numbers. As we go along the line, we will have in equation 2.82 the two terms moving in and out of phase, the resultant voltage magnitude having the form shown in Figure 2.24.

Note there are *two* maxima in each wave length. The maximum and minimum values of the voltage magnitude are

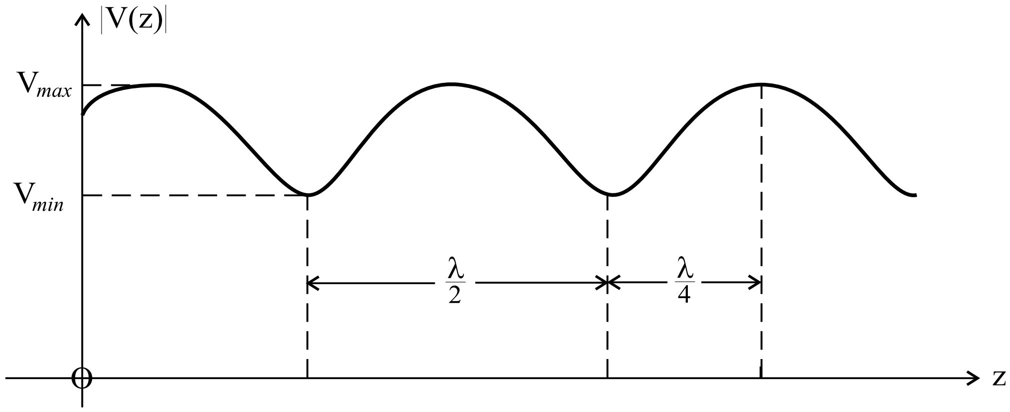


Figure 2.24: Variation of total voltage along a line.

$$\begin{aligned} V_{max} &= |V_f| + |V_r| \\ \text{and} \qquad V_{min} &= |V_f| - |V_r| \end{aligned} \tag{2.83}$$

Note that the incident wave will always be greater in magnitude than the reflected wave whenever the reflected wave comes from a passive termination, so that the last equation always produces a positive value.

2.10.2 Voltage standing wave ratio

A quantity of considerable practical importance is the Voltage Standing Wave Ratio (VSWR) denoted by S.

$$S = \frac{V_{max}}{V_{min}} = \frac{|V_f| + |V_r|}{|V_f| - |V_r|}. \tag{2.84}$$

If we divide by $|V_f|$ and obtain the relations

$$\boxed{|\Gamma_v| = \frac{S-1}{S+1}} \tag{2.85}$$

and

$$\boxed{S = \frac{1+|\Gamma_v|}{1-|\Gamma_v|}}. \tag{2.86}$$

The voltage standing wave ratio is one of the primary quantities determined from a slotted line measurement; the other is the position of a minimum of the standing wave pattern.

The impedance conditions on the lines are specified by giving Γ_v (which involves two real numbers) at any point. We can get $|\Gamma_v|$ from the above equation for S, and we can get the phase of Γ_v by noting that it will be real and *positive* at the voltage *maxima*, and real and *negative* at voltage *minima*.

Maximum and minimum values of impedance along the line can be related fairly simply to S. When $\mathrm{V}_f e^{-j\beta z}$ is in phase with $\mathrm{V}_r e^{+j\beta z}$ we have a simultaneous voltage maximum and current minimum. Thus

$$\boxed{Z_{max} = \frac{\mathrm{V}_{max}}{\mathrm{I}_{min}} = \frac{|\mathrm{V}_f| + |\mathrm{V}_r|}{(|\mathrm{V}_f| - |\mathrm{V}_r|)/Z_0} = SZ_0} \tag{2.87}$$

$$\boxed{Z_{min} = \frac{\mathrm{V}_{min}}{\mathrm{I}_{max}} = \frac{|\mathrm{V}_f| - |\mathrm{V}_r|}{(|\mathrm{V}_f| + |\mathrm{V}_r|)/Z_0} = \frac{Z_0}{S}} \tag{2.88}$$

As often happens, it is useful to translate these equations into words in order to aid the process of committing them to memory. The first equation indicates that the maximum impedance on the line is equal to the characteristic impedance times the standing wave ratio, while the minimum impedance on the line is equal to the characteristic impedance divided by the standing wave ratio.

We will take up the question of standing wave ratio again in the next chapter is the context of the study of the Smith Chart.

2.11 Calculation of Line Parameters

2.11.1 Laws we can use

When a structure has a significant degree of symmetry, it is usually possible to guess the shape of the field distribution, and obtain its funtional form from Ampère's Integral Law

$$\oint \mathsf{H} \cdot \mathbf{dr} = I \tag{2.89}$$

or Gauss' Integral Law

$$\oint \mathsf{D} \cdot \mathbf{ds} = Q. \tag{2.90}$$

In the calculations suggested to be done below, we will make the assumption mentioned in Chapter 1 that the charges reside on the surfaces of the conductors and the currents also flow on those surfaces.

2.11.2 Some important structures

An illustation of some important transmission line stuctures is provided in Figure 2.25. One of them, the *screened pair stucture*, contains more than two conductors (three in fact) and falls outside the scope of the this course.

2.11.3 Application to coaxial cables

Show as an *exercise* that for the coaxial cable of dimensions shown in Figure 2.26 the inductance per unit length and capacitance per unit length are given by

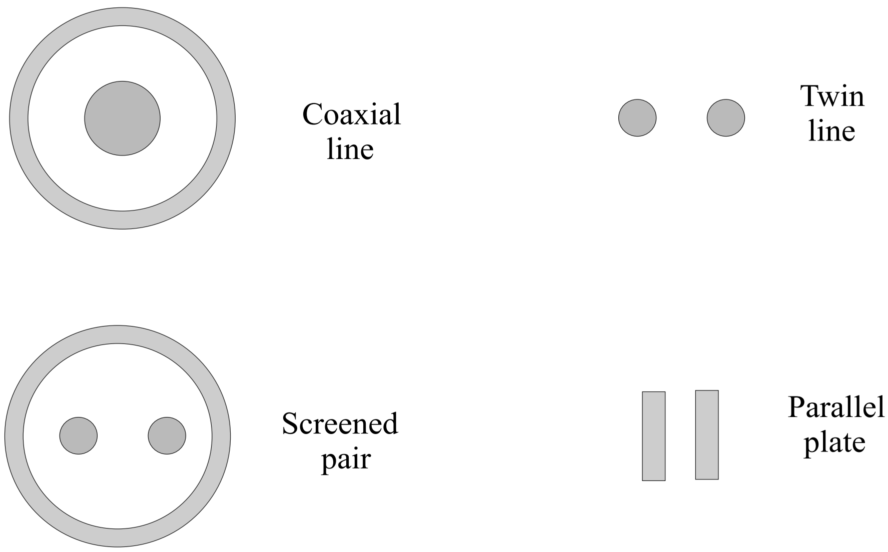


Figure 2.25: Various transmission line structures.

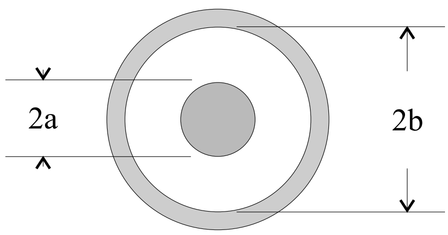


Figure 2.26: Dimensions of a coaxial cable.

$$L = \frac{\mu_0}{2\pi}\log_e\left(\frac{b}{a}\right) \tag{2.91}$$

$$C = \frac{2\pi\epsilon}{\log_e\left(\frac{b}{a}\right)}. \tag{2.92}$$

Students who cannot produce for themselves these results should consult the lecturer for help. From the above results we derive that the characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon}}\log_e\left(\frac{b}{a}\right) \tag{2.93}$$

Note this value never gets far in practice from 50Ω. To see this we rearrange the above equation into the form

$$Z_0 = \frac{1}{\sqrt{\epsilon}}\frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\log_e\left(\frac{b}{a}\right) \tag{2.94}$$

We note that the the second pair of factors amounts to $\eta/(2\pi)$ which has a value of approximately 60 ohms. Most of the practical dielectric materials for the making of coaxial transmission lines have relative dielectric permittivities of the order of 2.25 of which the square root is 1.5, so the first factor is about (2/3). The logarithm function is commonly greater than unity but is a slowly varying function of its argument. Hence the overall function never deprts much from 50 ohms. Values from 50 to 90 ohms are practical to maunfacture, but values outside this range are diffiult to achieve with common materials and geometries.

The complex propagation constant is

$$\alpha + j\beta = \sqrt{YZ} = j\omega\sqrt{LC} \tag{2.95}$$

This shows that firstly there is no attenuation, not surprising in the light of the fact that we assume there are no losses, and that the velocity $c = \omega/\beta$ is

$$\boxed{c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}}. \tag{2.96}$$

This result is *independent of frequency*, and equal to the velocity of light in the medium of parameters μ and ε. Common values of ε_r are 1 and about 2.25.

Losses R and G in coaxial lines we will discuss in a later chapter; however, we say now that R depends on frequency and G is generally negligible.

The reason for asserting that the resistance depends on frequency is closely related to our original assumption that the currents flow on the surface, but can only be fully explained in Chapter 9.

2.11.4 Twin lines

We state without proof that for twin lines of the type illustrated in Figure 2.27

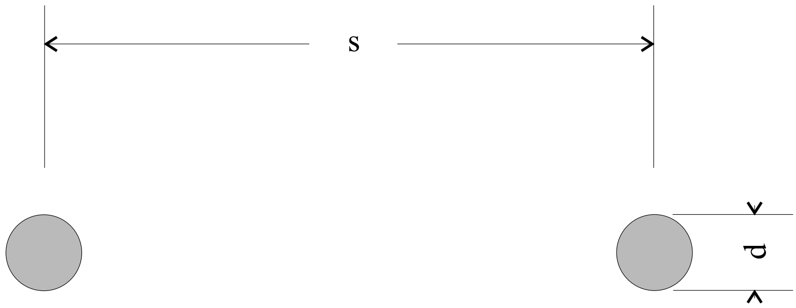


Figure 2.27: Dimensions of a twin line.

$$L = \frac{\mu_0}{\pi}\operatorname{arcosh}\left(\frac{s}{d}\right) \tag{2.97}$$

$$\approx \frac{\mu_0}{\pi}\log_e\left(\frac{2s}{d}\right) \qquad \text{when } s \gg d \tag{2.98}$$

$$C = \frac{\pi\varepsilon}{\operatorname{arcosh}\left(\frac{s}{d}\right)} \tag{2.99}$$

$$\approx \frac{\pi\varepsilon}{\log_e\left(\frac{2s}{d}\right)} \tag{2.100}$$

Again we note that the velocity is independent of frequency, and equal to velocity of light in the medium. The characteristic impedance is

$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\varepsilon}}\log_e\left(\frac{2s}{d}\right) \qquad \text{for } s \gg d \tag{2.101}$$

Common values of Z_0 are 300Ω for communication lines, 600Ω for telephone lines; slightly higher values are found for power lines.

2.11.5 More complicated structures

Useful tables of results of transmission line parameters for various further transmission line structures are found in Reference 1. In one edition of that book, Table 8.09 opposite page 444 and Table 1.25 opposite page 26 are of interest.

Chapter 3

MATCHING OF TRANSMISSION LINES

3.1 Introduction

In this chapter we will continue the development of transmission line theory in the important context of *transforming impedances* so that source impedances can be conjugately matched to load impedances. We will introduce the important concept of the Smith Chart which is an invaluable tool for the *visualisation* of forward and backward waves and of impedance conditions on a transmission line.

In all of the work in this chapter the transmission lines will be assumed to be lossless.

3.1.1 Meaning of matching

To give meaning to the word *matching*, we recall from lumped circuit theory the *maximum power transfer theorem* for a.c. circuits which indicates that a sinusoidal steady state source of fixed internal voltage V_S and source impedance Z_S will deliver maximum power to a load impedance Z_L when Z_L is adjusted to be the *complex conjugate* of the source impedance Z_S, that is

$$Z_L = Z_S^*. \tag{3.1}$$

3.1.2 Reasons for matching

Matching of the transmission line *at both ends* makes power transfer between the source and the load take place at minimum loss, and also makes the system behaviour become *independent of the line length.*

3.1.3 Method of matching

When matching is performed in a transmission line context it is done in the way illustrated in Figure 3.1. In this figure two matching systems are illustrated. The matching system on the right is intended to transform the load impedance Z_L so that it becomes equal to the characteristic impedance Z_0 of the transmission line. When that is done the input impedance of the transmission line with its transformed load is simply equal to

Z_0. The matching system on the left has the function of transforming the characteristic impedance of the transmission line Z_0 further to become the complex conjugate of the source impedance Z_S. In both cases the matching systems consist of purely *reactive elements* so no power is lost within them.

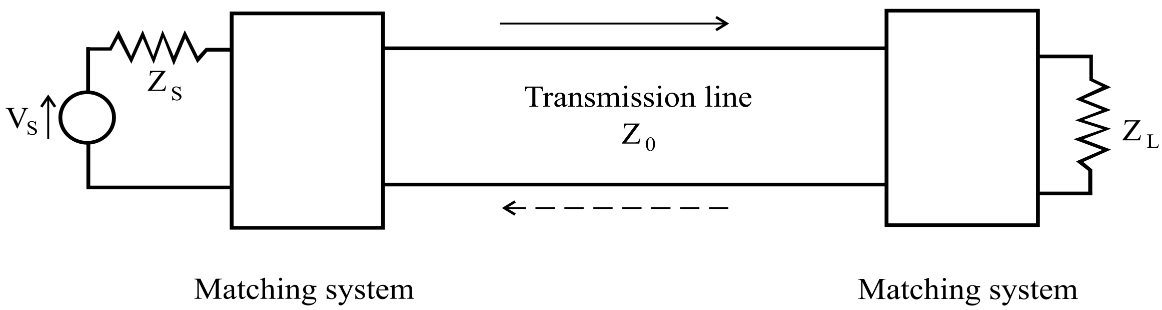


Figure 3.1: Matching at ends of a transmisison line.

It may be shown, but we will not do so here, that the same matching system will transform the source impedance Z_S into the real impedance Z_0, so that from either of two possible points of view maximum power transfer occurs between the source and the input end of the transmission line.

The matching systems often consist of short lengths of transmission line with reactive elements connected in series or in parallel.

3.2 Definition of the Smith Chart

The Smith Chart provides a graphical method for the *visualisation* of conditions on a transmission line and for the *solution* of transmission line problems. The method rests on the fact that the voltage reflection factor Γ_v defines the conditions on the line, and that as we move along a lossless line from the load to the source, the voltage reflection factor Γ_v transforms in a simple way, viz.

$$\Gamma_v(S) = \Gamma_v(L)e^{-2j\beta l} \tag{3.2}$$

This equation tells us that, as we move along the line in the direction of the generator by a length l, the voltage reflection factor maintains constant magnitude and changes in phase by an aount of $2\beta l$. This transformation has a simple construction in the complex plane illustrated in the Argand diagram of Figure 3.2.

We now provide the basic definition of the Smith Chart. Close attention should be paid to this definition, which should be committed to memory. This point is given some emphasis, because by the time the Smith Chart has been fully developed, many students have lost sight of its origin, and are thus handicapped in their understanding of it.

$\boxed{\text{\textit{The Smith Chart is a plot of } } \Gamma_v \text{\textit{ in the complex plane, with some extra information added.}}}$

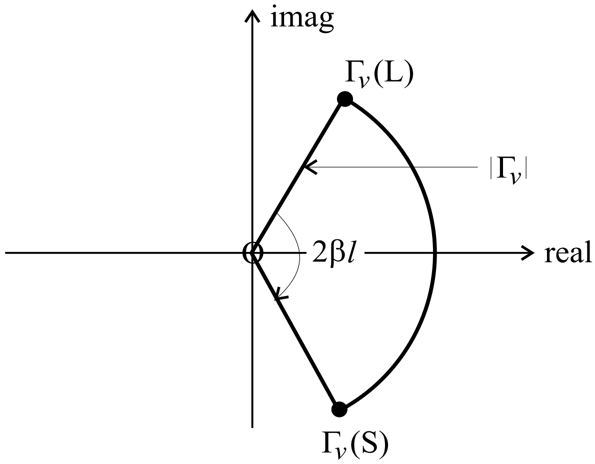


Figure 3.2: Argand diagram.

A skeletal form of a Smith chart is illustrated in Figure 3.3. Although we will do so later, for the purpose of developing theory, it is not usual to put in any Cartesian axes because we think of the voltage reflection factor simply in terms of the magnitude in phase terms illustrated in that figure.

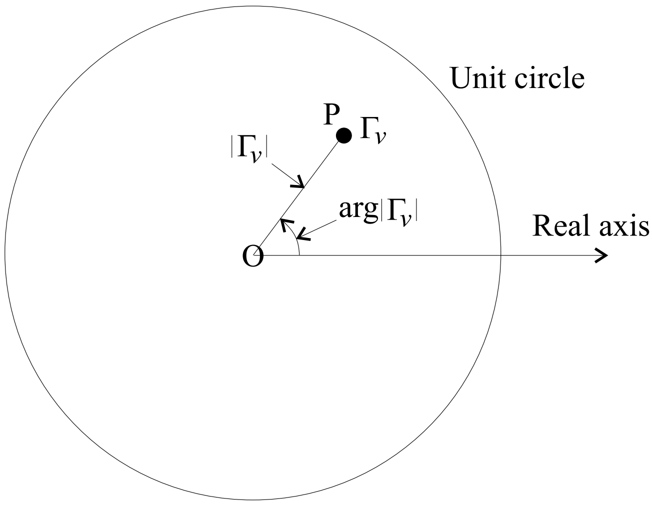


Figure 3.3: Skeletal form of a Smith Chart.

3.3 Elementary properties of the Smith Chart

We will now develop a number of important properties of the Smith Chart, the first of which is that for lines with passive terminations, the chart is confined to the interior (and the edge) of the unit circle.

For any impedance with positive real part, we will show that $|\Gamma_v| \leq 1$. To do this, let $z = r + jx$ be the *normalized impedance* at any point, then

$$
\begin{aligned}
\Gamma_v &= \frac{z-1}{z+1} = \frac{r+jx-1}{r+jx+1} \\
\text{so} \qquad |\Gamma_v|^2 &= \frac{(r-1)^2+x^2}{(r+1)^2+x^2} \\
\text{so} \qquad |\Gamma_v|^2 - 1 &= \frac{-4r}{r^2+x^2+2r+1}
\end{aligned}
$$

which, for passive impedances which must have positive values of r, is negative. Thus, the Smith Chart for passive terminations lies *within the unit circle*.

Now we ask what other information might usefully be put on a Smith Chart? Since there is a one-to-one correspondence between z and Γ_v, we can put values of $z = r + jx$ on the chart in the form of a *curvilinear grid.* To find out what these curves look like, we will, for the moment, insert after all a Cartesian co-ordinate system for Γ_v so that we can employ familiar results of Cartesian co-ordinate geometry. The results of doing this are illustrated in Figure 3.4. Thus we put

$$
\begin{aligned}
z &= r + jx \\
\text{and} \qquad \Gamma_v &= u + jv
\end{aligned}
\tag{3.3}
$$

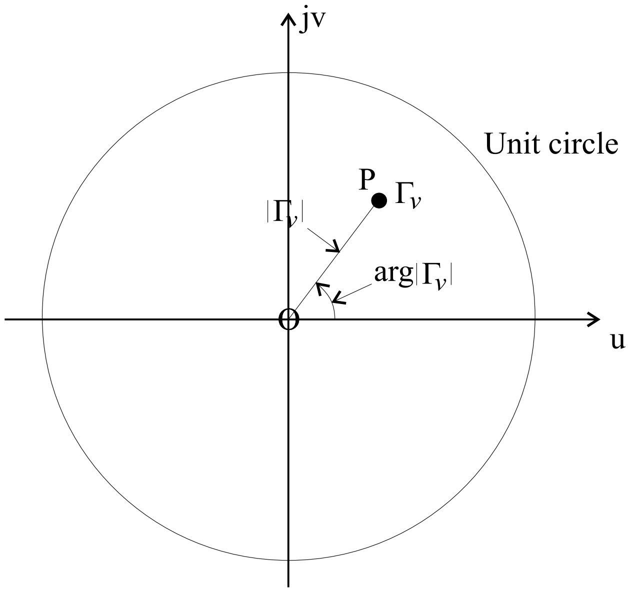


Figure 3.4: Cartesian axes temporarily placed on a Smith Chart.

Now we ask: What is the nature of the curve where r is fixed and x varies? From the basic $z - \Gamma_v$ relation, we have

$$
z = r + jx = \frac{1+\Gamma_v}{1-\Gamma_v} = \frac{1+u+jv}{1-u-jv}
$$

We may separate the right hand side into real and imaginary parts, and derive

$$r = \frac{1-(u^2+v^2)}{(1-u)^2+v^2} \tag{3.4}$$

$$x = \frac{2v}{(1-u)^2+v^2} \tag{3.5}$$

We can rearrange 3.4 to obtain

$$\left(u - \frac{r}{1+r}\right)^2 + v^2 = \frac{1}{(1+r)^2} \tag{3.6}$$

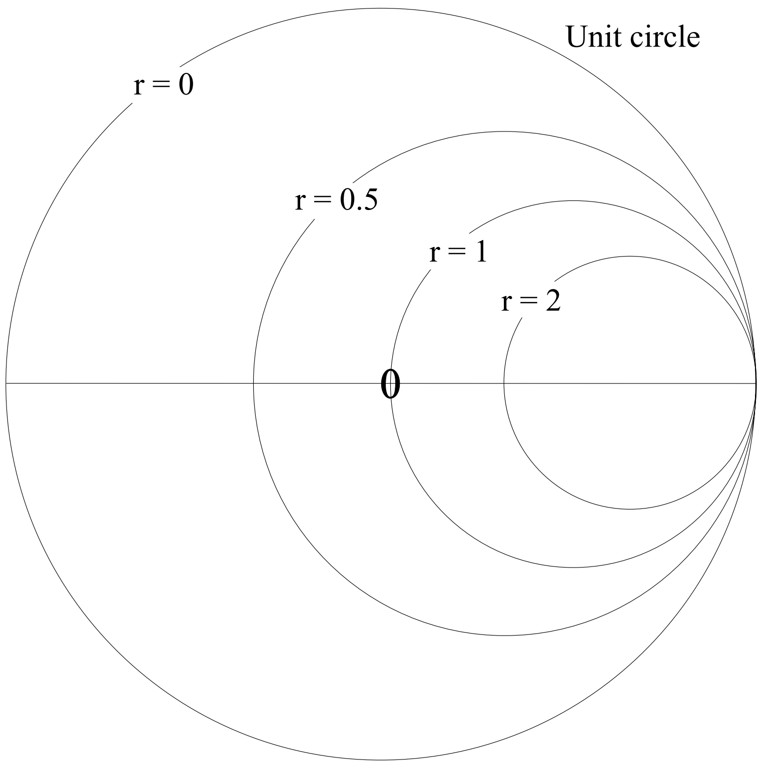


Figure 3.5: Constant resistance circles on a Smith Chart.

This is the *equation of a circle*, centre $(r/(1+r), 0)$ and radius $1/(1+r)$. The circle always passes through the point (1,0). Hence, we see the constant r circles are as shown in Figure 3.5.

By similar analysis, we find the constant x curves are circles of centre $(1, 1/x)$ and radius $1/x$. They also all pass through the point $(1, 0)$.

These have the form shown in Figure 3.6; we draw only the parts of them which lie within the unit circle.

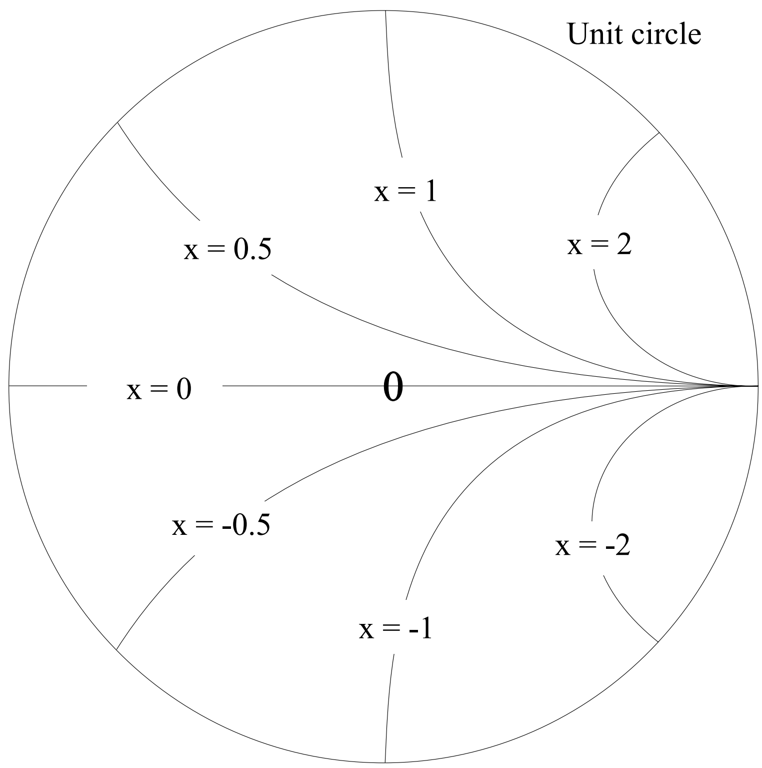


Figure 3.6: Constant reactance circles on a Smith Chart.

3.4 Applications of the Smith Chart

3.4.1 Use for $\Gamma_{\mathsf{v}} - z$ or $\Gamma_{\mathsf{i}} - y$ relations

We have so far developed the Smith Chart as a graphical expression of the voltage reflection factor Γ_v with impedance contours placed upon it, but the similarity between the relations between the voltage reflection factor and the normalised impedance and between the current reflection factor and normalised admittance allows the chart to be interpreted equally well as an expression of current reflection factor with normalised admittance contours placed upon it. Thus a single chart can be used either as a $\Gamma_v - z$ or a $\Gamma_i - y$ chart.

We have already noted that the equation $\Gamma_i = -\Gamma_v$ showing the relation between current and voltage reflection factors. Thus the connection between Γ_v and Γ_i is by *reflection in the origin.* Thus, we can get y from z, and vice versa, by reflection in the origin, and using the same set of curves as contours of normalised resistance r and reactnce x or as contours of normalised conductance g and susceptance b.

3.4.2 Transfer along a line

To find the input impedance of a line of length l and characteristic impedance Z_0 terminated in a load impedance Z_L, we normalise Z_L with respect to Z_0 to obtain z_L, enter the Smith Chart at z_L, move *clockwise* by an amount $2\beta l$ radians, and read off the normalised input impedance z_{in} at the point resulting. The unnormalized input impedance is $Z_{in} = Z_0 z_{in}$.

To make such operations simple, the Smith Chart has a peripheral scale graduated in λ, thus avoiding the need for conversion of line lengths to radians. Note that movement of only *half a wave length* along a line corresponds to movement of *one full revolution* around the chart.

3.4.3 Voltage standing wave ratio

We recall from Chapter 2 the results

$$S = \frac{R_{max}}{Z_0} = r_{max} \tag{3.7}$$

$$S = \frac{Z_0}{R_{min}} = \frac{1}{r_{min}} \tag{3.8}$$

and the fact that the point of maximum voltage on the line occurs as the point of maximum impedance and the point of minimum voltage on the line occurs at the point of minimum impedance, and that those points are distant a quarter of a wave length apart. These matters are illustrated in Figure 3.7 which shows the locus of a voltage reflection factor, which originates from a point at which the normalised load impedance is z_L, as we move back along the line away from the load.

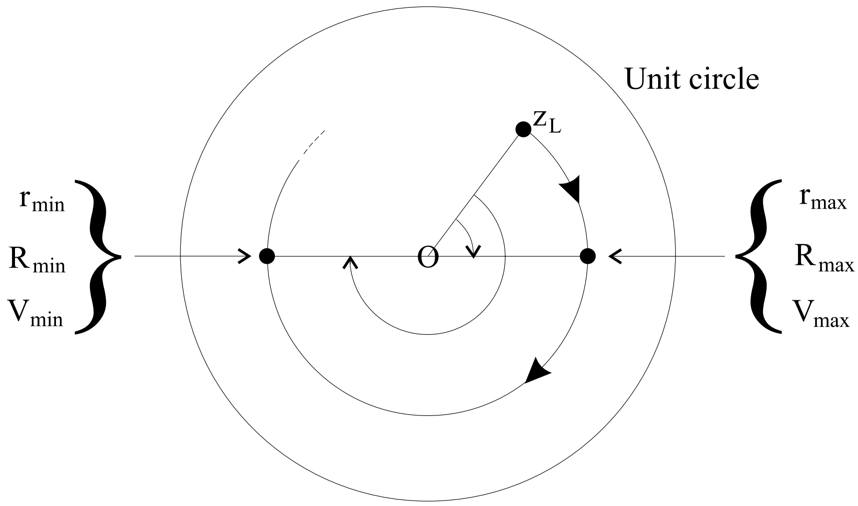


Figure 3.7: Points of maximum and minimum voltage.

When that locus intersects the horizontal axis to the right of the origin, the voltage reflection factor has become real and positive, and maximum voltage and impedance occur at that point. Moving further along the line away from the load produces a second intersection of the horizontal axis, this one to the left of the origin, at which point the voltage reflection factor has become real and negative, and the point of minimum voltage and impedance is encountered. We see that the movement from the point z_L towards the input end of the line passes through points of voltage maximum and minimum on the line, as shown in Figure 3.7.

3.4.4 Design of stubs

Practical reasons, will be discussed in lectures, make it more convenient in many transmission line problems to work in *admittance* terms. Thus when we design a stub, we are seeking to find the stub length which will provide a particular *susceptance*.

We recall from Section 2.8.2 and Section 2.9.2 that we can obtain any desired susceptance by using either a short-circuit or open-circuit stub of suitable length. Again, practical reasons suggest variable stubs should be *short circuited*.

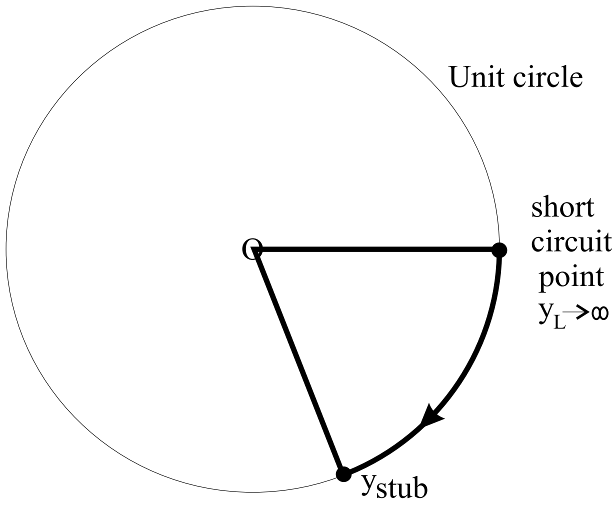


Figure 3.8: Finding lengths of short circuited stubs.

The length of a short circuited stub to produce a desired susceptance B may be obtained from the formulae in Table 2.3, or by using the Smith Chart as in Figure 3.8. In this figure, to produce a desired input susceptance B we first of all find the normalised input admittance y_{stub} of the stub corresponding to the susceptance B by dividing the admittance $0 + jB$ by the characteristic admittance of the transmission line forming the stub. If we are using a short circuited stub, the load end on the admittance chart is the right-most extremity of the unit circle, as illustrated in Figure 3.8. The correct stub length is found by moving *clockwise* from that point away from the load (considered to be the short circuit) towards the input end of the stub until the desired normalised input admittance y_{stub} is found.

To obtain the stub length from the angular rotation we can make use of the fact that a *complete revolution* around the Smith chart corresponds to movement along the line by a length of *one half of a wave length*, or we can use a peripheral scale on the Smith Chart which is specifically provided to assist in this calculation.

The rotation from the start point shown in Figure 3.8 is always in the direction shown. In the example illustrated in that figure the susceptance of the stub is negative, and the rotation is less that half a revolution, and thus the stub length will be less than a quarter of a wave length.

When the susceptance required for the stub is positive, a point on the upper half of the unit circle must be reached, and a rotation of more than half a revolution is required, and the length of the stub is more than a quarer of a wave length.

Clearly stub lengths of more than half a wave length are never required.

3.5 Single Stub Matching

We will now show that, using a single shunt stub of variable length, and a section of transmission line of variable length, we can match any load impedance Z_L to Z_0, which for practical purposes could be the characteristic impedance of a transmission line designed to connect that load to a distant source.

3.5.1 Configuration

The matching system is shown in Figure 3.9. For the reasons described in a later paragraph we are working in *admittance* terms.

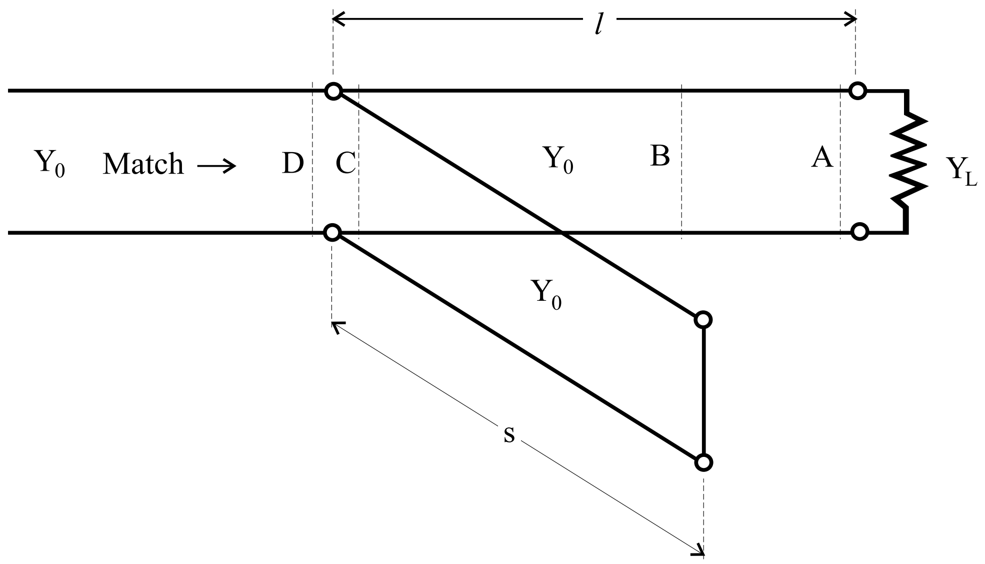


Figure 3.9: Line configuration for single stub matching.

At the right of the diagram is shown is shown a load admittance Y_L. A line of characteristic admittance Y_0 and length l connects the load to a junction point between the main line and a short circuited stub line of which the characteristic admittance is also Y_0. To the left of that junction, i.e. at the reference plane D, a transmission line whose characteristic impedance is also assumed to be Y_0 is supposed to be *matched*, i.e a wave propagating to the right in that line is not reflected when it reaches the junction. This will be because, when we have correctly designed the matching system, the admittance of the parallel combination of the admittance provided by the stub line and the admittance at the reference plane C is equal to the characteristic admittance Y_0 of the line at the reference plane D to the left of the junction.

In the design of this matching system, the lengths s and l are the variables to be found. Because the stub is connected *in parallel*, we work in *admittances*, which combine in a simple way (they simply add) for parallel connections.

3.5.2 Procedure

For single stub matching, Figure 3.10 gives an illustration on the Smith Chart of the procedure. We work in *normalized* admittances on the chart.

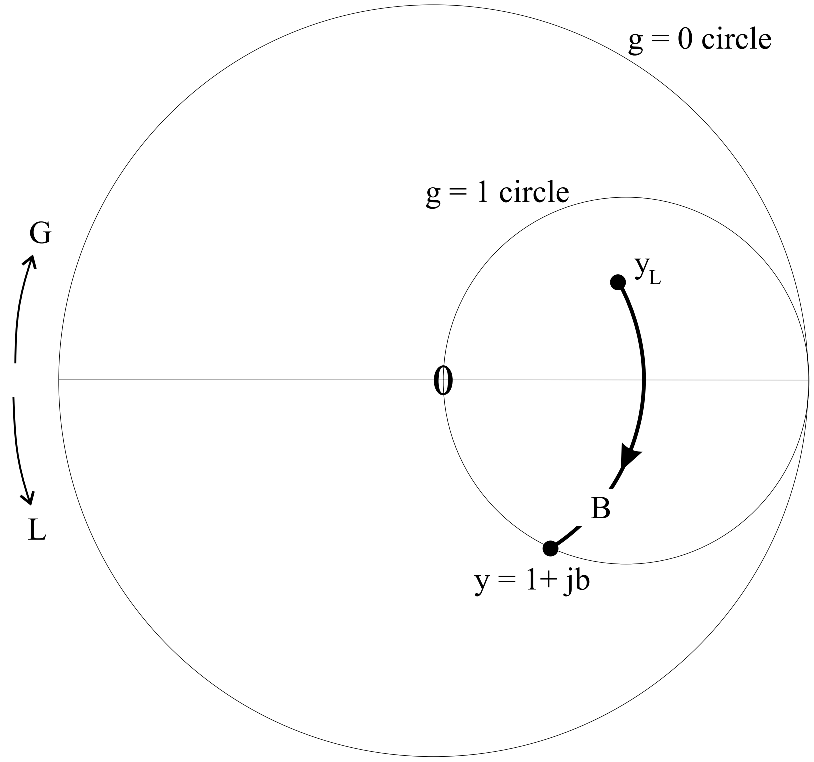


Figure 3.10: Transformation along a line in single stub matching.

The steps of the design procedure are as follows.

1. Normalize Y_L with respect to Y_0, and enter an admittance Smith Chart at the point y_L so found. This point will represent the normalised admittance seen looking to the right at the reference plane shown as A in Figure 3.9.

2. Draw the locus of the normalised admittance looking toward the load through various lengths of transmission line, i.e. at variously positioned reference planes B as shown in Figure 3.9. This locus will be the circle of centre the origin which passes through y_L. We denote this locus, which is shown in Figure 3.10, as locus B. We note that various values of normalised conductance g and normalised susceptance b are encountered as we move along the locus, and that the locus must at some point intersect the circle $g = 1$, in fact at two places.

   At either of these places we have achieved the desired normalised conductance $g = 1$, but we have an unwanted susceptance which we will call jb.

3. Read off the susceptance jb which results at that point of intersection. If we *remove* this susceptance by the parallel connection of a susceptance $-jb$ from a stub, we will the have achieved at the reference plane D the desired result of having a normalised conductance of unity, accompanied by zero susceptance.

4. The problem is now to find the length s of a stub of susceptance $-jb$. To do this we enter as shown in Figure 3.11 a second Smith Chart at the short circuit point $y \to \infty$ on the periphery and travel clockwise, i.e. towards the input end of the stub, until we get to the point $y = 0 - jb$.

   In the example given, the rotation to get from the short circuit point to the point $y = -jb$ is about seven eighths of a revolution, so the length of the short circuited stub will be about seven sixteenths of a wave length.

   To determine the actual length in metres of the stub we must know the wavelength on the stub line at the operating frequency.

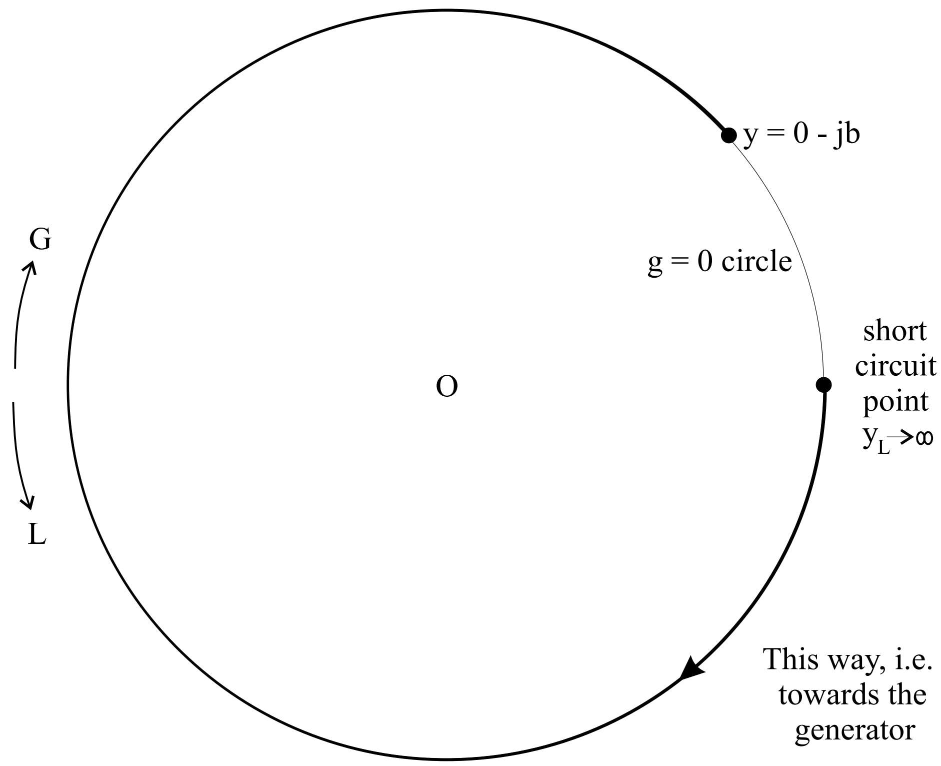


Figure 3.11: Determination of stub length in single stub matching.

5. We must also determine the length of the section of transmission line between the load and the stub. This length is determined by the amount of rotation which took place in Figure 3.10 to get from the load point y_L to the $g = 1$ circle, using the rule that rotation once around the Smith Chart corresponds to movement of half a wave length along transmission line.

   In the example given, the rotation appers to be a little less thatn a quarter of a revolution, so the line length l will be a little less than an eighth of a wave length.

   To determine the actual distance in metres moved along the line we must know the wavelength at the operating frequency.

3.5.3 Additional remarks

Some addtional remarks about single stub matching are set out below.

- We note that the procedure described above yields two separate solutions, of which we have only developed one, for the matching problem. There is no simple rule for preferring one solution to the other.
- One of the nice features of single stub matching is that the procedure always works, because wherever on the Smith Chart the normalised load admittance y_L lies, an intersection with the $g = 1$ circle is guaranteed when we move from that normalised admittance y_L at constant radius around the chart. In fact two such interesections always occur.
- In the above illustration of single stub matching, we have assumed that the line to the left of the reference plane D, the stub line of length s and the line of length l connected to the load, all have the same charactreistic admittance Y_0. In a more general matching problem, the three lines jut mentioned can have different characteristic admittances. Obvious adjustments to the procedure must then be made. Principally, these adjustments consist of transforming un-normalised admittances to normalised admittances or vice versa when one moves from a transmission line of one characteristic impedance to a transmission line of another characteristic impedance, because it is the un-normalised admittances which always add at a junction, but the normalisd admittances only do so when the normalising admittances have a common value.
- The fact that the procedure always works stands in contrast to the double stub matching method to be discussed in the next section, which works, in its simplest form, for some load admittances but not for others. As we will see, however, the adoption of a more complex form of double stub matching provides a cure for that problem.

3.6 Double Stub Matching

The construction of variable length lines involves the fabrication of low loss sliding contacts for both of the conductors, and as a result variable length lines are expensive. It is thus more economical to use two variable-length stubs and a fixed-length line for matching.

3.6.1 Configuration

In the normal arrangement, the stubs are separated by a distance of $L = 3\lambda/8$ as shown in Figure 3.12, but other separations are possible, with appropriate variations to the procedure described below.

In Figure 3.12 we see that the load admittance Y_L and a stub known as S2 are both connected to the right hand end of a length $3\lambda/8$ of line. At the left hand end of that line an additional stub known as S1 is connected in parallel. To the left of that junction the line is supposed to be *matched*, that is waves travelling rightward in that leftmost section of the line experience no reflection when they encounter the junction just to the right of the reference plane D.

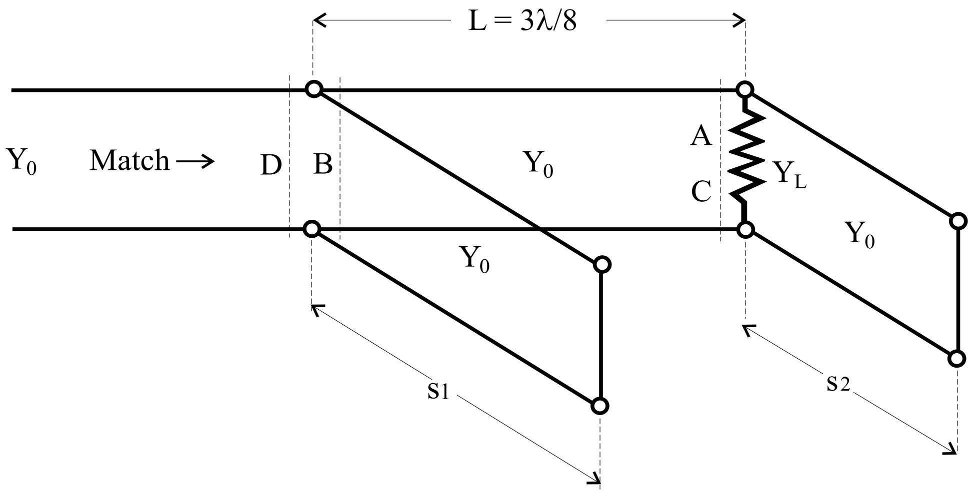


Figure 3.12: Line configuration for double stub matching.

3.6.2 Simple description

Before we discuss the steps of the procedure, some general remarks are in order. The first is that the solution procedure for double stub matching is more complex and more indirect that has just been outlined for single stub matching.

One way to describe the procedure is that, referring to Figure 3.12, we assemble partial information about the solution in two stages. Firstly we will work at the *load end* of the tuner, and express what information we can about the solution as a locus of points on the Smith Chart. This locus has the property that the final solution lies somewhere on it, but we do not know where. This locus corresponds to the partial solution as seen at the reference plane indicated as A on the diagram of Figure 3.12.

Then we transfer our attention to the *source end* of the tuner, and again express partial information about the solution as another locus on the Smith Chart. Again we know that the final solution must lie on that locus, but we do not know where. This locus corresponds to the partial solution as seen at the reference plane indicated as B on the diagram of Figure 3.12.

The core of the solution procedure is to transform the locus corresponding to the reference plane B back to the reference plane A at the load end of the tuner to thus derive a new locus C. We then have, at the reference plane which has been labelled as both A and C, two different loci drawn on the Smith Chart on which loci we know the solution must lie. The solution therefore lies at their point of intersection.

3.6.3 Detail of the procedure

The details of the procedure are illustrated in Figure 3.13 and the discussion below.

As indicated in Figure 3.12, the length $3\lambda/8$ length of line, the two stubs and the main line to be matched all have a characteristic admittance Y_0. This is the normal case but the procedure can be varied to allow for cases where the characteristic admittances of those four line lengths do not have a common value. Such variation will be discussed

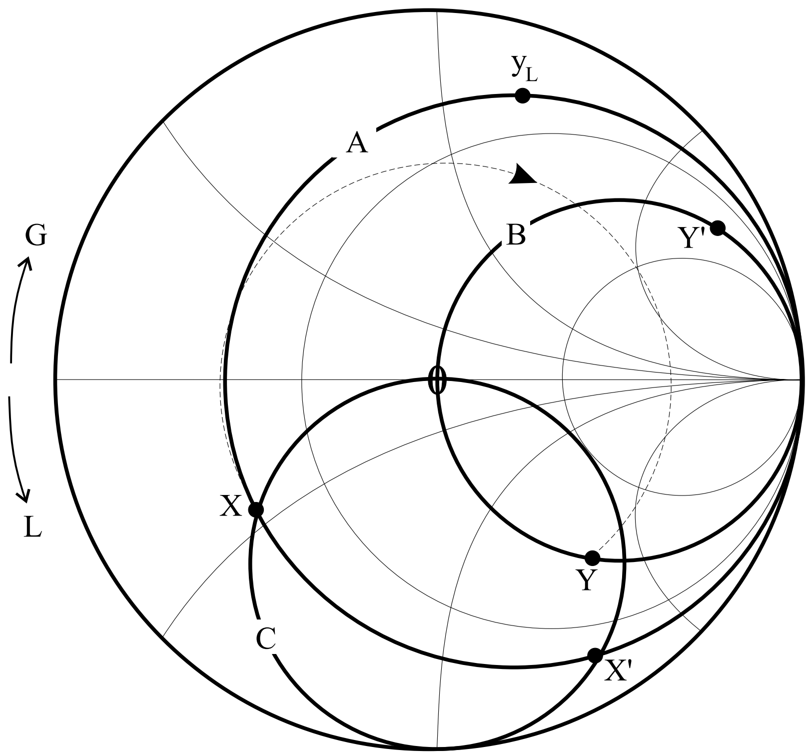


Figure 3.13: Operations for double stub matching with $L = 3\lambda/8$

later.

The steps of the procedure, all of which are in terms of normalized admittance, are as follows.

1. Locate the normalised load admittance y_L on the chart. In the example shown in Figure 3.13 this point is shown towards the top of the chart.

2. Now we take note of the fact that the action of connecting stub S2 in parallel with the load admittance Y_L will be to produce a resultant admittance at the reference plane A, just to the left of the junction between stub S2 and the load, which has the same normalised conductance as the load but has a normalised susceptance which is at present unknown because we do not know the length of the stub S2, and therefore do not know the value of the susceptance provided by it. Thus we can say that the admittance looking to the right at the reference plane A has the same normalised conductance as the load and an unknown normalised susceptance. Thus that normalised admittance lies on the constant g circle which passes through the point y_L. We call this locus A, and draw it on the Smith Chart.

   Locus A represents a partial solution to the problem in that when we do find a solution, it must lie on that locus.

3. We for the moment put aside the information that we have obtained by working from the load end of the tuner and is expressed in locus A, and transfer our attention to the input end of the tuner, and examine what conclusions we can draw about the normalsied admittance at the reference plane shown as D in Figure 3.12.

4. Firstly we note at the reference plane D our desire is that the transmission line lying to the left of that point be *matched*. Thus we know that the normalised admittance corresponding to the reference plane D has the values $g = 1$ and $b = 0$. This corresponds to the single point which is shown as the origin of the Smith chart, i.e. at its centre.

5. Next we consider what conclusions we can draw about admittances at the reference plane B which is just to the right of the junction between the main line, the tuner line, and stub S1. At this reference plane B, we can say that, because the parallel connection of the stub can only change the susceptance, the normalised conductance is the same as that at the reference plane D, but we do not know the normalised susceptance at the reference plane B because we do not know the susceptance provided by stub S1. Given that the normalised conductance at reference plane D is $g = 1$, we can say that the noralised admittance at reference plane B lies on the $g = 1$ circle. The set of points which form the $g = 1$ circle have been shown as locus B on the diagram of Figure 3.13.

6. We are now in the position of having locus A representing some of the information about the solution and locus B representing different information about the solution. We might be tempted to simply look for an intersection between locus A and locus B to find a point which is consistent with both pieces of information, but to do so would be incorrect, because locus A expresses information which is valid at the right hand end of the tuner, i.e at reference plane A, and locus B represents information which is valid at the left hand end of the tuner, i.e. at the different reference plane B.

7. What we need to do is to take the information that we have contained in locus B and re-express it as a different set of points which correspond to the same information but expressed at the position of the reference plane at the load end of the tuner, i.e. at the reference plane which has been labelled as both A and C, and which is distant $3\lambda/8$ from reference plane B in the direction of the load.

   To do this we take locus B as a solid object, and rotate it by three quarters of a revolution in a *counter-clockwise* direction, the fixed point of the rotation being the centre of the Smith chart. This leads us to the locus shown at C in Figure 3.13.

8. We now have partial information about the solution in locus A and partial information about the solution in locus C, and both locus A and locus C express their information at a common reference plane, namely that labelled as both A and C in Figure 3.12. It is now true that the point of intersection of locus A and locus C defines a solution to the problem.

9. Thus *two* solutions, at points X and X' can be found at the intersection of locus A and locus C. We select *one* of these solutions, at the point X, for the further development below.

10. We are now in a position to determine the normalised susceptance to be provided by stub S2, and hence the length s_2 of stub S2.

11. To do this we observe that the function of stub S2 is to modify the normalised susceptance at the reference plane denoted as A and C so that the total normalised susceptance b_X at that point consists of the sum of the normalised load susceptance b_L and the normalised susceptance b_{s2} of stub s2.

12. Thus we calculate the normalised susceptance to be provided by stub S2 as $b_{S2} = b_X - b_L$, both of the quantities on the right hand side of this equation being readable from the Smith Chart.

13. We then calculate from the desired normalised susceptance the length s_2 of stub S2 on a separate chart, using the procedure which was defined in Section 3.4.4, and which was further described in the context of single stub matching in Section 3.5.

14. Although we have now found the length of the stub S2, we have further work to do to find the length of the stub S1. We cannot use either of the solution points X or X' to directly find the stub length, because those solution points apply at the reference plane labelled as A and C, and stub S1 is connected to the line at a different point, namely at the reference plane B, which is three eighths of a wave length further away from the load. So what we must do is to re-express our chosen solution point X or X' so that it applies at the reference plane B just to the right of the junction between stub S1 and the main line.

    At that different reference plane the solution points X or X' become different points Y or Y' on the Smith Chart. To find those different points Y or Y' we rotate the solution points X or X' by three quarters of a wave length in a *clockwise* direction, i.e. towards the generator. Quite naturally those solution points are found to lie on locus B, because we already knew that the solution as expressed at reference plane B would lie on locus B.

15. Let us assume that our chosen solution point is X on locus A and C and that it leads to point Y on locus B. At that point Y we have the desired normalised conductance $g = 1$, and a susceptance b_Y which can be read from the Chart.

16. To find the normalised susceptance to be provided by the stub S1 we note that what we accomplish by the parallel connection of stub S1 at a point between reference planes B and D is the *removal* of the susceptance b_Y which is still present at reference plane B, but is not wanted. Thus what we want from stub S1 is an input susceptance of $-b_Y$.

17. The method of calculating the length s_1 of stub S1 to provide that now known susceptance has already been defined in Section 3.4.4, and was further described in the context of single stub matching in Section 3.5.

3.6.4 Additional remarks

Some additional remarks on the procedure now follow.

- The stubs and the main line can be of different characteristic impedances. We should then use three different charts, and work in normalized admittances on each.

But to transfer values from one chart to another, e.g. transferring b_{S2} and b_{S1} to get s_1 and s_2, we must first un-normalize using the characteristic admittance of the main line, and then re-normalize using the characteristic admittance of the stub line.

- Not all values of admittance can be matched by the above tuner. As an *example*, we examine the *forbidden region* for load admittances on the Smith Chart. We find the result is the *interior* of the $g = 2$ circle.

- In such cases, we can insert a $\lambda/4$ section of transmission line between the load and the tuner, and in so doing transform the load admittance to a value which is well outside the forbidden region.

3.7 Exercises

1. Find for the usual arrangement of stub separation of $3\lambda/8$ the region of normalised load admittances on the Smith Chart which cannot be matched by the double stub tuner described above. This region is called the *forbidden region*, and its complement the *allowable region*.

2. Show that the insertion of a $\lambda/4$ section of transmission line between the load and a normally configured (i.e. $3\lambda/8$) double stub tuner will move a normalised admittance from the forbidden region into the allowable region on the Smith Chart.

Chapter 4

TIME VARYING ELECTROMAGNETIC FIELDS

4.1 Conservation of Charge Concept

As our first major concept we begin with the notation that there is a thing called charge which is conserved. It can be stationary or in motion. It can be used to probe an electromagnetic field; it is discrete but in such small amounts as not to concern us at the macroscopic level. The charge and its motion are characterised by parameters

DESCRIPTOR	SYMBOL	UNITS
Charge density per unit volume	ρ	Cm^{-3}
Volume current density	J	Am^{-2}
Surface current density	K	Am^{-1}

Table 4.1: Important charge and current density descriptors.

such that each of the expressions in Figure 4.1 gives the amount of charge crossing the indicated boundary per unit time, i.e. the current.

4.1.1 The conservation equation

The equation expressing the fact that charge cannot be created or destroyed, but can merely be moved around in the form of an electric current, which if of suitable non-uniformity in space might cause a change of charge density to arise, is in integral form

$$\boxed{\frac{\partial}{\partial t}\int_v \rho dv = -\oint_S \mathsf{J} \cdot \mathbf{ds}} \tag{4.1}$$

or in differential form

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathsf{J}}. \tag{4.2}$$

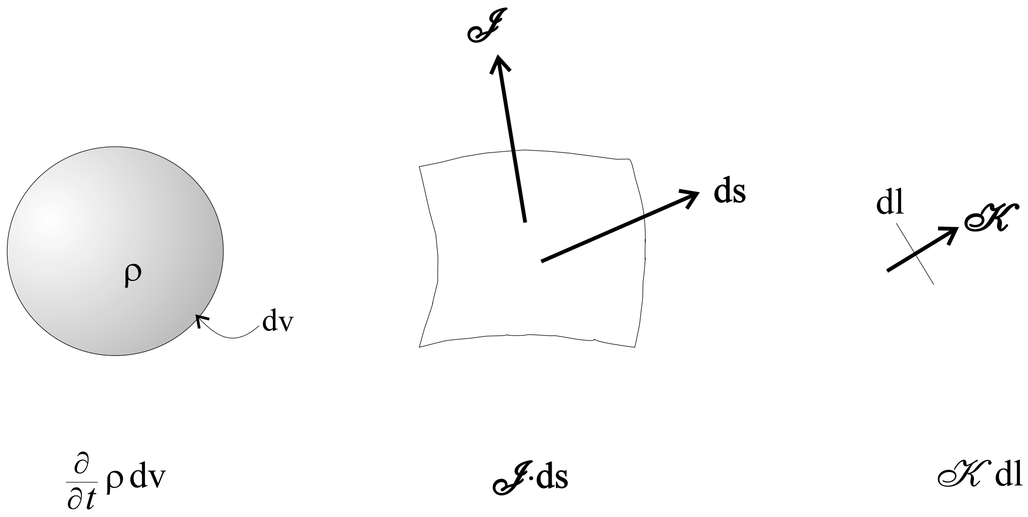


Figure 4.1: Illustration of charge and its motion.

4.2 Force on Moving Charge

The force F F on a charge q moving *in a vacuum* in which the electric and magnetic fields are E and H respectively is given by

$$\boxed{\mathsf{F} = q\left(\mathsf{E} + \mu_0 \mathbf{v} \times \mathsf{H}\right)} \tag{4.3}$$

This equation in effect defines E and H in SI units. The corresponding SI units of a number of quantities are as follows.

SYMBOL	UNITS	
F	newton	N
ρ	coulomb	C
E	volt/metre	V/m
μ_0	henry/metre	H/m
v	metres/s	m/s
H	ampere/metre	A/m

Table 4.2: Important quantities and their SI units.

4.3 The Field Equations in Free Space

4.3.1 Differential form

In *free space* the laws of electrodynamics take the following *differential form*. They were first correctly formulated by J.C. Maxwell, and are known as Maxwell's equations.

$$\begin{aligned}
\nabla \times \mathsf{E} &= -\frac{\partial(\mu_0 \mathsf{H})}{\partial t} \\
\nabla \times \mathsf{H} &= \mathsf{J} + \frac{\partial(\epsilon_0 \mathsf{E})}{\partial t} \\
\nabla \cdot (\epsilon_0 \mathsf{E}) &= \rho \\
\nabla \cdot (\mu_0 \mathsf{H}) &= 0
\end{aligned} \tag{4.4}$$

These equations imply Faraday's law of induction, Coulomb's law for a static electric field, Ampères law as modified by Maxwell for production of magnetic field either by conduction currents or by "displacement currents", and the fact that we as yet have found no isolated magnetic poles.

4.3.2 Integral form

Difficulties sometimes arise in using the differential equations when boundaries are introduced and idealized, and some fields become discontinuous at the boundary. In such cases the *integral forms* of the equations shown below are useful. The medium is still free space.

$$\begin{aligned}
\oint_C \mathsf{E} \cdot \mathbf{dr} &= -\int_S \frac{\partial(\mu_0 \mathsf{H})}{\partial t} \cdot \mathbf{ds} \\
\oint_C \mathsf{H} \cdot \mathbf{dr} &= \int_S \mathsf{J} \cdot \mathbf{ds} + \int_S \frac{\partial(\epsilon_0 \mathsf{E})}{\partial t} \cdot \mathbf{ds} \\
\oint_S (\epsilon_0 \mathsf{E}) \cdot \mathbf{ds} &= \int_v \rho \, dv \\
\oint_S (\mu_0 \mathsf{H}) \cdot \mathbf{ds} &= 0
\end{aligned} \tag{4.5}$$

These laws are also particularly convenient when geometrical symmetry of a structure allows us to make a plausible assumption about the shape of the electromagnetic field distribution, and reduces the number of unknown components of the field which must be found to a small number.

4.4 The Field Equations in the Presence of Media

4.4.1 Point of view

The point of view we adopt regards matter as atomic, with internal charges and electric and magnetic dipoles capable of acting as sources of electromagnetic field, and also of being influenced by the total field set up.

4.4.2 Electric effects

The electric effects can be described on a basis of *separation of internal charges*, to give rise to a *polarization vector*

$$\mathsf{P} = N\mathbf{p} \tag{4.6}$$

described as the result of N dipoles, each of strength $\mathbf{p}$, created per unit volume. The *charge dq crossing an area* $\mathbf{ds}$ in the direction of $\mathbf{ds}$ in the process of establishing the polarisation is

$$dq = \mathsf{P} \cdot \mathbf{ds}. \tag{4.7}$$

The *induced surface charge density* at a boundary between a polarised and unpolarised medium is a charge density per unit area of

$$\rho_s^i = \mathsf{P} \cdot \hat{\mathbf{n}}. \tag{4.8}$$

The *induced volume charge density* which results when P is non-uniform, is

$$\rho^i = -\nabla \cdot \mathsf{P}. \tag{4.9}$$

The *induced volume current density* which results when the polarisation is changing with time is given by

$$\mathsf{J}^{\,i} = \frac{\partial \mathsf{P}}{\partial t}. \tag{4.10}$$

The *induced* charges and currents act as sources for the electromagnetic field in just the same way as *conduction* charges and current, and they can be inserted into the second and third equations of 4.4 to give

$$\begin{aligned} \nabla \cdot (\epsilon_0 \mathsf{E}) &= \rho^c + \rho^i && (4.11)\\ \nabla \times \mathsf{H} &= \mathsf{J}^{\,c} + \mathsf{J}^{\,i} + \frac{\partial(\epsilon_0\,\mathsf{E})}{\partial t}. && (4.12) \end{aligned}$$

Because of the simple relations of 4.9 and 4.10 which relate the induced charges and currents to P, we have

$$\begin{aligned} \nabla \cdot (\epsilon_0\,\mathsf{E}) &= \rho^c - \nabla \cdot \mathsf{P} && (4.13)\\ \nabla \times \mathsf{H} &= \mathsf{J}^{\,c} + \frac{\partial \mathsf{P}}{\partial t} + \frac{\partial(\epsilon_0\,\mathsf{E})}{\partial t}. && (4.14) \end{aligned}$$

Introducing the vector

$$\text{electric flux density } \mathsf{D} = \epsilon_0 \mathsf{E} + \mathsf{P} \tag{4.15}$$

we may write the above equations in the more compact form

$$\begin{aligned} \nabla \cdot \mathsf{D} &= \rho^c && (4.16)\\ \nabla \times \mathsf{H} &= \mathsf{J}^{\,c} + \frac{\partial \mathsf{D}}{\partial t}. && (4.17) \end{aligned}$$

4.4.3 Magnetic effects

The magnetic effects are caused mainly by the spin of the electrons of which the matter is composed, sometimes by the orbital motion of the electrons in the atoms, and sometimes by the spin or motion of the other particles. The fields are however always describable by giving a magnetic medium a *density per unit volume of magnetic dipole moment* which we call M and which gives rise to magnetic fields in the same way as P gives rise to electric fields. Thus the first and fourth Maxwell's equations are modified to

$$\begin{aligned} \nabla \cdot \mathsf{H} &= 0 - \nabla \cdot \mathsf{M} && (4.18) \\ \nabla \times \mathsf{E} &= -\frac{\partial(\mu_0 \mathsf{H})}{\partial t} - \frac{\partial(\mu_0 \mathsf{M})}{\partial t}. && (4.19) \end{aligned}$$

Introducing the vector

$$\text{magnetic Flux Density } \mathsf{B} = \mu_0(\mathsf{H} + \mathsf{M}) \tag{4.20}$$

the above equations can be written in the more compact form

$$\begin{aligned} \nabla \cdot \mathsf{B} &= 0 && (4.21) \\ \nabla \times \mathsf{E} &= -\frac{\partial \mathsf{B}}{\partial t}. && (4.22) \end{aligned}$$

4.4.4 Summary

In summary, the differential form of Maxwell's equations in the presence of media is

$$\begin{aligned} \nabla \times \mathsf{E} &= -\frac{\partial \mathsf{B}}{\partial t} \\ \nabla \times \mathsf{H} &= \mathsf{J} + \frac{\partial \mathsf{D}}{\partial t} \\ \nabla \cdot \mathsf{D} &= \rho \\ \nabla \cdot \mathsf{B} &= 0. && (4.23) \end{aligned}$$

In the integral form which we may obtain by the use of Gauss' and Stokes' theorems, Maxwell's equations become

$$\begin{aligned} \oint_C \mathsf{E} \cdot \mathbf{dr} &= -\int_S \frac{\partial \mathsf{B}}{\partial t} \cdot \mathbf{ds} \\ \oint_C \mathsf{H} \cdot \mathbf{dr} &= \int_S \mathsf{J} \cdot \mathbf{ds} + \int_S \frac{\partial \mathsf{D}}{\partial t} \cdot \mathbf{ds} \\ \oint_S \mathsf{D} \cdot \mathbf{ds} &= \int_v \rho \, dv \\ \oint_S \mathsf{B} \cdot \mathbf{ds} &= 0. && (4.24) \end{aligned}$$

These equations, together with the definitions 4.15 and 4.20, are regarded as the basic laws of electrodynamics in the presence of media. Note that they are quite general in that they do not assume linearity or spatial uniformity of those media.

Because these equations have introduced additional variables, they no longer contain sufficient information for the solution to practical problems. Before a solution to a practical problem can be obtained we need additional information. Such additional information comes from specifying the relation between E and P (or E and D) and between H and M (or H and B) within these media.

The specification of E-P and H-M relations for various classes of media is the subject of the next chapter.

Chapter 5

BEHAVIOUR OF MATERIALS

5.1 Objective

Practical electromagnetic theory problems almost always involve material media and/or finite boundaries. To solve such problems we need firstly a knowledge of how material media react to the internal electromagnetic fields within them, and a knowledge of electromagnetic boundary conditions. It is the objective of this chapter to give a simple treatment of the first topic, and of the next chapter to give a comprehensive treatment of the latter.

5.2 Constitutive Parameters

5.2.1 Introduction

Having generalised in Section 4.4 Maxwell's Equations to take into account the effects of polarisation and magnetisation within media, we find we are left with too few equations in the number of unknowns now involved, to obtain a solution. The number of equations in the expanded set of variables is made sufficient when we provide a specification of the relationship between the polarisation and magnetisation vectors P and M and the internal electric and magnetic fields E and H which may be regarded as their direct cause.

In our physical experience there is a wide variety of media which we might encounter, and hence a variety in the behaviour to be described. There is also a variation in the degree of precision with which it might be convenient to describe that behaviour. The subject we have identified is therefore a wide one, and we will provide here only an elementary and incomplete treatment.

5.2.2 Linear lossless dielectrics

In most non-crystalline dielectrics, at least for electric field strengths below that of dielectric breakdown, the polarisation is proportional to *and in the direction* of the internal electric field. This is expressed by the equation

$$\mathsf{P} = \chi_e\, \epsilon_0\, \mathsf{E} \tag{5.1}$$

where χ_e is a dimensionless constant called the *dielectric susceptibility*. From the above equation it follows that the electric flux density D is given by

$$\mathsf{D} = (1 + \chi_e)\, \epsilon_0\, \mathsf{E} \tag{5.2}$$

An alternative expression of this result is

$$\mathsf{D} = \epsilon_r\, \epsilon_0\, \mathsf{E} \tag{5.3}$$

where $\epsilon_r = (1+\chi_e)$ is a further dimensionless parameter known as the *relative dielectric permittivity*, or alternatively as the *dielectric constant*. Alternatively we may write

$$\mathsf{D} = \epsilon\, \mathsf{E} \tag{5.4}$$

where ϵ is called the *dielectric permittivity* and has the same units as ϵ_0, i.e. units of Fm^{-1}. It may be shown that with this type of dielectric behaviour, no losses are involved in changing the state of polarisation of the material. Models of dielectric media suitable for a wide variety of purposes can be obtained by combining the above lossless D - E relation with the linear conductivity relation, which does involve energy loss, and is defined in Section 5.2.11 below.

5.2.3 Non-linear but lossless dielectric

A more general form of behaviour, which admits of some non-linearity in the material, but still does not involve energy loss, is described by the equation

$$\mathsf{P} = \chi_e(E)\, \epsilon_0\, \mathsf{E} \tag{5.5}$$

where $\chi_e(E)$ is now a dimensionless dielectric susceptibility which depends on the magnitude of the electric field strength. This model is of theoretical use in establishing certain energy conservation theorems, but is not much needed for description of practical materials as dielectrics are generally highly linear until breakdown is reached.

5.2.4 Linear crystalline dielectric

If a dielectric medium is in the form of a single crystal, it will generally exhibit different polarisation responses to electric fields in different directions. This behaviour is described by the equation

$$\mathsf{P} = \epsilon_0\, \underset{\approx}{\chi} \cdot \mathsf{E} \tag{5.6}$$

where $\underset{\approx}{\chi}$ is called the dimensionless dielectric susceptibility tensor, whose components are specified by a 3×3 array.

This model is useful in that it does represent the behaviour of practical single-crystal dielectric media. We will not in later sections of this chapter introduce a corresponding model for single-crystal magnetic media, because inter-atomic interactions in magnetic media are quite different from those in dielectric media, with the result that single-crystal magnetic media behave in the main in an entirely different way.

5.2.5 Permanently polarised ferroelectrics

In some dielectric media the atoms possess permanent electric dipole moments which can, once permanently aligned by the application of a strong electric field, remain in that condition unless other strong electric fields are applied. This behaviour is described by the equation

$$\mathsf{P} = \mathsf{P}_0 \tag{5.7}$$

where P_0 is as constant vector.

5.2.6 Linear lossy dielectric

In some dielectric materials, losses involved in changing the state of polarisation cannot be entirely neglected. In these materials, we can sometimes write a linear differential equation relating the polarisation and its time rate of change, and the corresponding quantities for the electric field. Specification of the amount of energy lost would be possible if we knew the details of the time variation involved.

When the time variation is *sinusoidal*, the results admit of a simple description. For a sinusoidally varying electric field, the resulting polarisation response is also sinusoidal, but lags a little behind the electric field. This behaviour is described by the phasor equation

$$\mathbf{P} = (\chi_e' - j\chi_e'')\,\epsilon_0\,\mathbf{E} \tag{5.8}$$

where χ_e' and χ_e'' are the energy-storage and energy-loss components of a *complex dielectric susceptibility.* Please note that although losses are involved, the response is still linear in that the phasor $\mathbf{P}$ is still proportional to the phasor $\mathbf{E}$. The corresponding description in terms of a complex dielectric permittivity is

$$\mathbf{D} = (\epsilon' - j\epsilon'')\,\mathbf{E} \tag{5.9}$$

It might be noted, by comparing the form of this equation with that of the electrical conductivity equation appearing is Section 5.2.11, that although the physical mechanisms might be regarded as being different, the effects of electronic conduction loss and polarisation loss are, on a macroscopic scale and with sinusoidal excitation, indistinguishable.

5.2.7 Linear soft ferromagnet

In *polycrystalline ferromagnets* and for *small values* of internal field strength we may write to a reasonable level of approximation

$$\mathsf{M} = \chi_m\,\mathsf{H} \tag{5.10}$$

where χ_m is a dimensionless parameter called the *magnetic susceptibility.* The same relation may be expressed in the alternative form

$$\mathsf{B} = \mu_0(1 + \chi_m)\,\mathsf{H} \tag{5.11}$$

Introducing the *relative magnetic permeability* $\mu_r = (1 + \chi_m)$ we may write

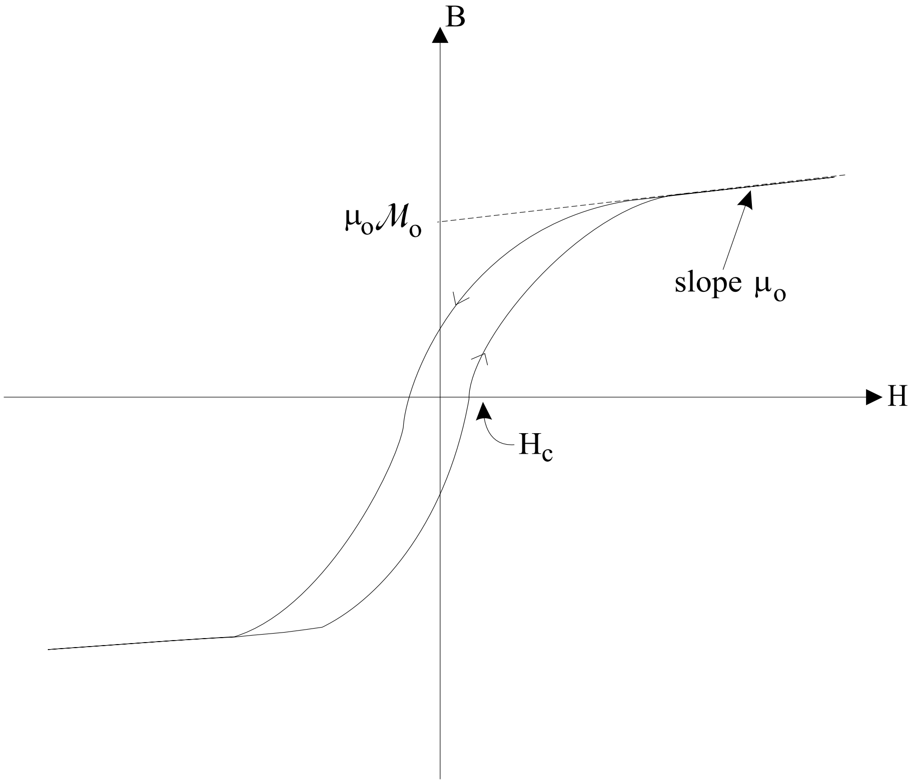


Figure 5.1: An hysteresis curve.

$$\mathsf{B} = \mu_r \mu_0 \, \mathsf{H} \tag{5.12}$$

ie

$$\mathsf{B} = \mu \, \mathsf{H} \tag{5.13}$$

where μ is called the *magnetic permeability* and has the same units as μ_0 i.e. has units of Hm^{-1}.

5.2.8 Non-linear but lossless ferromagnet

For larger field strengths, the behaviour of a ferromagnet is better described by the *hysteresis curve* shown in Figure 5.1. In relation to that curve we take the opportunity to define in the figure the important concepts of *saturation magnetisation* M_0 and *coercive force* H_c.

If the hysteresis loop is relatively narrow, and we are prepared to accept a rather crude approximation, we could write

$$\mathsf{M} = \chi_m(H) \, \mathsf{H} \tag{5.14}$$

where $\chi_m(H)$ is the field-dependent non-linear magnetic susceptibility. This model corresponds to the dielectric model of equation 5.5, and has approximately the same relatively uninteresting uses.

5.2.9 Linear lossy ferromagnet

If we are prepared to consider *small signals*, and the *sinusoidal steady state*, we can model the behaviour of a material in which the magnetisation lags somewhat behind a sinusoidal magnetic field excitation by the equations

$$\mathbf{M} = (\chi_m' - j\chi_m'') \, \mathbf{H} \tag{5.15}$$

$$\mathbf{B} = (\mu' - j\mu'') \, \mathbf{H} \tag{5.16}$$

The linear lossy ferromagnet model is tantamount to replacing the hysteresis curve of Figure 5.1 by an ellipse. Although the model does not exhibit the saturation phenomenon shown in that figure, at least the energy loss property of the hysteresis curve is retained.

5.2.10 Saturated ferromagnet

In a *dielectric* medium the individual atoms may or may not have permanent dipole moments. When an electric field is supplied, atoms initially without a dipole moment may acquire one with a strength proportional to the electric field, and the polarisation described in equation 5.1 results.

In the case of dielectric media in which the atoms already have, in the absence of an internal electric field, a dipole moment, thermal agitation normally ensures that these are randomly aligned, and in the absence of an electric field, no polarisation is macroscopically evident. When an electric field is applied a partial alignment, proportional to the strength of the internal field, of the dipole moments occurs, and again the macroscopically observed polarisation obeys the relation described in equation 5.1.

The situation in a ferromagnet is quite different. In a *ferromagnetic* medium, the *individual atoms* possess *permanent magnetic moments*, which are fixed in magnitude although not in direction, but a phenomenon of quantum mechanical origin and known as the *exchange effect* causes the dipole moments of several hundred to many thousands of adjacent atoms to remain mutally exactly aligned. The result is that such groups of atoms are arranged in what are known as *domains*, which are *internally permanently and fully magnetised*. In the absence of an applied field, there is no magnetisation evident on a macroscopic scale because the orientation of the magnetisation in different domains of the material is randomly distributed.

When magnetic fields are applied to the material, the domain wall boundaries move to cause the enlargement of domains with a magnetisation direction similar to that of the applied field at the expense of domains in which the magnetisation is oppositely directed. How far the domain wall boundaries move is dependent upon the strength of the applied field, with the result that the magnetisation relations equation 5.10 and 5.14 are on a macroscopic scale observed. When however the applied field is made strong enough, the material becomes magnetised as a *single domain* in which the magnetisation has the same direction as the applied field and has a value equal to the magnetic moment of an individual atom multiplied by the number of atoms per unit volume. This terminal behaviour is described by the equations

$$|\,\mathsf{M}\,| = M_0 \tag{5.17}$$

$$\mathsf{M} \text{ is directed along } < \mathsf{H}_0 > \tag{5.18}$$

where M_0 is called the *saturation magnetisation* of the material, H_0 is the externally applied field, and the $< >$ symbol represents the time average value. No further increase in the magnitude $|\,\mathsf{M}\,|$ of the magnetisation is then possible. This behaviour has *little resemblance to that of dielectric media*.

What happens when *time-varying* magnetic fields of small amplitude are then superimposed upon the large steady magnetic field H_0 normally required for saturation bears even less resemblance to the corresponding behaviour in dielectric media. To understand what will happen, we must take note of a coupling which is known to exist at the atomic level between the *angular momentum* of the electrons in the material and the *magnetic moments* of those electrons. This coupling is responsible for the *gyromagnetic behaviour* discussed below.

If time-varying magnetic fields are applied in a saturated ferromagnet in a direction transverse to the magnetisation, they have the effect of producing on the magnetic moments of the electrons in the magnetised atoms a torque. This torque will produce a time rate of change of angular momentum of the electrons in a direction which is perpendicular both to the applied time-varying field and to the saturation magnetisation. The result is that the magnetisation will take up a *precessional motion* in which its magnitude does not change, its time average value is still along the large magnetising field H_0 producing the original saturation, but its instantaneous direction will be at some angle to that field.

Although a mathematical description of this behaviour is well beyond the scope of this course, its occurrence has been included as an illustration of the wide variety of material responses to the electromagnetic field which is present in nature; and so that the phenomenon, which will receive detailed description in later years of the course, will not then be too unexpected.

5.2.11 Linear conductor

In many media the free electric charges can move in response to the internal electric field. Over a wide range, the forces opposing that motion are proportional to the *drift velocities* of the charge carriers. The result is the linear conduction relation

$$\mathsf{J} = \sigma\, \mathsf{E} \tag{5.19}$$

which is the expression of Ohm's law for volume current density flowing in a three dimensional region.

Since there appear to be no free magnetic charges, there is no magnetic analogue of this equation.

5.2.12 Comments

The discussion of constitutive parameters has preceded to the present length, largely to provide preparation for electromagnetic courses to be given in later years. For the purposes of the present course, only the linear lossless dielectric, linear soft ferromagnet and linear conductivity relations of sections 5.2.2, 5.2.7 and 5.2.11 need to be known in detail.

Chapter 6

ELECTROMAGNETIC BOUNDARY CONDITIONS

6.1 Introduction

Practical electromagnetic theory problems almost always involve material media and/or finite boundaries. To solve such problems we need firstly a knowledge of how material media react to the internal electromagnetic fields within them, and a knowledge of electromagnetic boundary conditions. It is the objective of this chapter to give a comprehensive treatment of the latter topic.

Practical electromagnetic theory problems almost invariably also involve both finite geometries, and discontinuities between the parameters characterising the medium in one section and those pertaining in another.

In a formal sense electromagnetic boundary conditions are required so that solutions to Maxwell's equations in differential form, which solutions involve the usual arbitrary constants, may be suitably *matched* as we cross such boundaries.

In a less formal sense, our knowledge of electromagnetic boundary conditions is required for another purpose. Any thorough understanding of electromagnetic theory must be based on a series of mental pictures of the possible electromagnetic field configurations which can occur in various geometries. Our knowledge of the requirements on electromagnetic field components at various plane boundaries, and in particular at metallic boundaries, is necessary for firstly the *visualisation* and secondly the *validity checking* of such potentially correct field pictures.

It might be said that the source and vortex interpretation of Maxwell's equations in differential form, and a knowledge of the shortly to be derived results on boundary conditions, is with experience sufficient in most cases for the construction of a qualitatively correct field solution without detailed mathematical investigation.

6.2 Boundary Characterisation

We will for simplicity consider *plane boundaries*, and will regard any smoothly curved boundary as approximately plane at an appropriate scale of viewing. Such a plane boundary is shown in Figure 6.1 and is assumed to lie between region 1, in which the medium is characterized by real magnetic permeability μ_1, dielectric permittivity ϵ_1, and electric

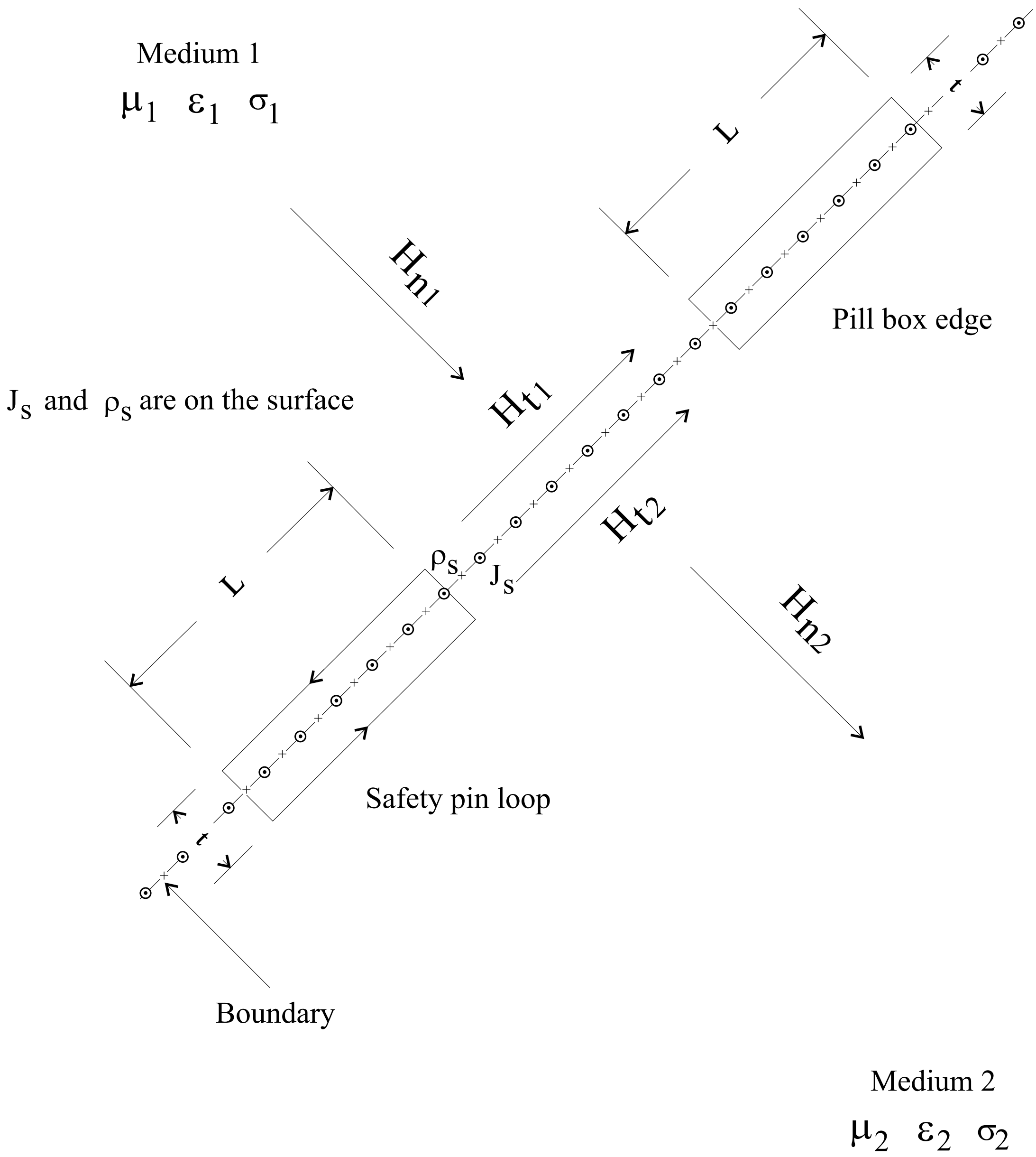


Figure 6.1: Variables and contours used in establishing electromagnetic boundary conditions.

conductivity σ_1, and a region 2 in which the material is characterized by corresponding parameters μ_2, ϵ_2, and σ_2. A unit vector $\hat{\mathbf{n}}$ is directed normal to the boundary from region 1 to region 2. The reference directions for E, D and B are similar to those for H which are shown.

The boundary is assumed to carry a possible surface charge density ρ_s, and a possible surface current density J_s which should be viewed as directed *out of the paper*.

In order to make a correct interpretation of some of the results to be derived below, particular relationships between the quantities just defined and the *reference directions* for them as established in Figure 6.1 must be observed. It should be noted that the reference direction for the surface current is *out of the paper.* The safety pin loop to which the line integrals will be applied lies with its long sides both in the plane of the boundary and in the plane of the paper, that is perpendicular to the direction of the surface current. Moreover the direction of traverse of that contour, which is indicated by the arrows in Figure 6.1 and the reference direction for the surface current, are related according to the *right-hand rule*. Finally, we note that the reference directions for tangential components on each side of the boundary will on one side match the direction of traverse of the contour but on the other side will be opposite. We will later take note of this fact in establishing the signs of terms which appear in the equations to be derived.

6.3 Method of Analysis

Because of the discontinuities in the material parameters, and the discontinuities in at least some components of the electromagnetic field which result therefrom, Maxwell's equations in differential form fail, in the sense that the derivatives do not exist, at the boundary. Maxwell's equations in integral form, however, still apply, as such finite discontinuities are readily integrable.

Maxwell's equations in integral form involve two contour integrals and two surface integrals. The method of analysis is to apply those equations to particular integrals over special contours or surfaces appropriately chosen in relation to the boundary. For the *line integrals*, the chosen contour is a *safety pin loop* of length L and thickness t of which one long side lies on each side of the boundary. This loop is shown in Figure 6.1.

In the case of the *surface integrals*, the chosen surface is the *pill box surface.* In the pill box, the large flat surfaces are of dimensions $L \times L$ and lie parallel to and on each side of the boundary. The box has thickness t. In Figure 6.1 the large $L \times L$ surfaces are viewed from an edge, and appear as a line.

6.4 The General Case in the Time Domain

It is a simple matter, and should be taken as an *exercise*, to show that Maxwell's equations in integral form applied to these contours lead to the results

$$\mathsf{E}_{t2} - \mathsf{E}_{t1} = 0 \tag{6.1}$$

$$D_{n2} - D_{n1} = \rho_s \tag{6.2}$$

$$\hat{\mathbf{n}} \times (\mathsf{H}_{t2} - \mathsf{H}_{t1}) = \mathsf{J}_s \tag{6.3}$$

$$B_{n2} - B_{n1} = 0 \quad (6.4)$$

In the above equations the subscript t indicates a tangential component and subscript n a normal component of the relevant field. For those students who have difficulty with this derivation the details will be presented in lectures.

In the two centre equations, note must be taken of the order of the terms on the left hand side. In equation 6.2, the term which appears with the *positive sign* is that expressing the *outward* component of the normal vector from the medium in question, while the term appearing with the *negative sign* expresses an *inward* vector to the medium in question. In equation 6.3 the term which appears with the *positive sign* is that in which the reference direction established for the tangential field and the sense of the contour *match*, while the term for which the *negative sign* appears has its reference direction *opposite* to the sense with which the contour is to be traversed.

The results which have just been derived are the most general expression of electromagnetic boundary conditions and as such are always valid. In words they state that the tangential component of electric field intensity and the normal component of magnetic flux density are always continuous across a boundary, while the normal component of electric flux density and the tangential component of magnetic field intensity can suffer discontinuities if surface charges or surface currents are present respectively.

Whether such surface charges or currents can in fact be present is determined by particular properties of the media present on each side of the boundary, and will be discussed in particular cases below.

The general case which we have just discussed can be particularised in two directions. Firstly the quite arbitrary time dependence assumed so far could be replaced by either no time dependence, i.e. the electrostatic or magnetostatic situation, or by a sinusoidal steady state time dependence. In another aspect, the general case may be particularised by assuming particular values for the material parameters. For example materials may be idealised as perfect insulators, perfect conductors, or simply media with finite (including zero) conductivity. These particularisations can create a potentially large number of cases through which we will try to pick our way with care.

It will be convenient firstly to consider particularisations of the media, and with each of those cases then to consider particularisations of the time dependence.

6.5 Imperfect Conductors

6.5.1 Definition

We regard an imperfect conductor as a medium with a finite (including zero) conductivity. Insulators will under this definition be a sub-class of imperfect conductors.

When both media are imperfect conductors, we consider in the next two sections the possibilities of having a surface current or a surface charge density, and then examine in the following section the impact upon the boundary conditions.

6.5.2 Surface currents

A *surface current* can be regarded as the limiting case when a finite amount of current I flows as shown in Figure 6.2 in a thin slab of area dimensions a and b and thickness t, and we let $t \to 0$.

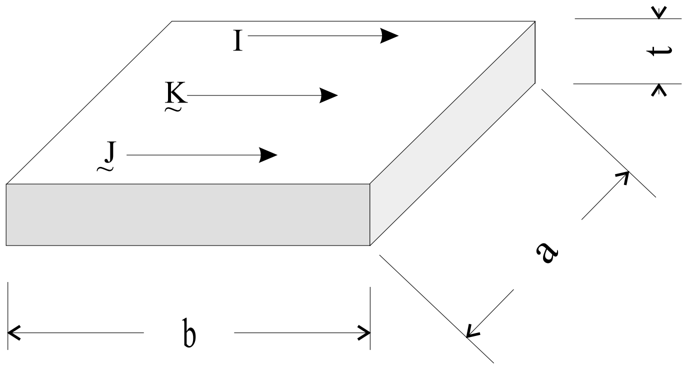


Figure 6.2: Conducting slab

While t is small but not zero we may describe this situation in terms of a volume current density J or a surface current density K. We may relate the magnituds of these quantities to I by

$$I = Ka = Jat. \tag{6.5}$$

If the material has electric resistivity ρ (note this symbol does not for the moment represent volume charge density) the resistance of the slab is

$$R = \frac{\rho\, b}{at}. \tag{6.6}$$

The power $P = I^2R$ dissipated in the slab is therefore given by

$$P = \frac{K^2a^2b\rho}{at} \tag{6.7}$$

i.e.

$$P = \frac{K^2ab\rho}{t}. \tag{6.8}$$

If K and ρ are non-zero, i.e. we have a surface current and the conductor is not perfect, then $P \to \infty$ as $t \to 0$. Since we cannot produce an infinite amount of power we must have $K = 0$, i.e. *we cannot have a surface current density in an imperfect conductor.*

6.5.3 Consequences at a boundary

The absence of a surface current thus reduces the boundary conditions to

$$\begin{aligned}
\mathsf{E}_{t2} - \mathsf{E}_{t1} &= 0 && (6.9)\\
D_{n2} - D_{n1} &= \rho_s && (6.10)\\
\hat{\mathbf{n}} \times (\mathsf{H}_{t2} - \mathsf{H}_{t1}) &= 0 && (6.11)\\
B_{n2} - B_{n1} &= 0. && (6.12)
\end{aligned}$$

These are not much changed from equations 6.1 to 6.4.

6.5.4 Possibility of a surface charge

We note in particular that a surface charge density ρ_s can exist, in many situations, even though not all.

It can, for example, exist as a d.c. value on the surface of any material, i.e. a perfect insulator, (after it has been rubbed with the cat), on a non-insulating imperfect conductor, or on a perfect conductor.

It can also exist as an a.c. value on the surface of a perfect conductor or an imperfect conductor, as even when surface currents are outlawed in the latter, volume currents directed perpendicular to the surface can make a surface charge density change.

The case when ρ_s is outlawed is when the fields and other variables are all sinusoidal and both materials are perfect insulators. There can in this situation be no mechanism of charge transport to change the surface charge density.

6.6 Two Insulating Media

The discussion above has made it clear that we cannot in this case have a surface current density J_s and neither can we have a time-varying surface charge density ρ_s. We can have an unvarying ρ_s.

6.7 One Perfect Conductor

6.7.1 Perfect conductor concept

We take a practical view of the term *perfect conductor* to mean a material with very high conductivity, such as a metal, within which any electric field must be negligibly small, but we can still establish a steady value of magnetic field H and magnetic flux density B, if we are prepared to spend a reasonable time doing it.

We are thus not discussing super-conducting media from which magnetic field are expelled through processes different from those considered in the course so far.

6.7.2 Possible interior fields

We are going for definiteness to make medium 1 the perfect conductor. In this medium there can be no electric field, either time-varying or static, and thus no electric flux density, either time-varying or static.

Because **curl** $\mathsf{E} = -\frac{\partial \mathsf{B}}{\partial t}$ there can be no time-varying magnetic flux density B and hence in this assumed linear medium, no time varying magnetic field H. There can however be a steady magnetic flux density B and magnetic field H. In practice these fields take a long time to establish or to change to a new steady value, the time depending on just how large the conductivity of the material is.

6.7.3 Consequences at a boundary

As said above we make medium 1 the perfect conductor. This is so that the vector $\hat{\mathbf{n}}$ points out into the adjacent space, as shown in Figure 6.3. For d.c. fields we would have

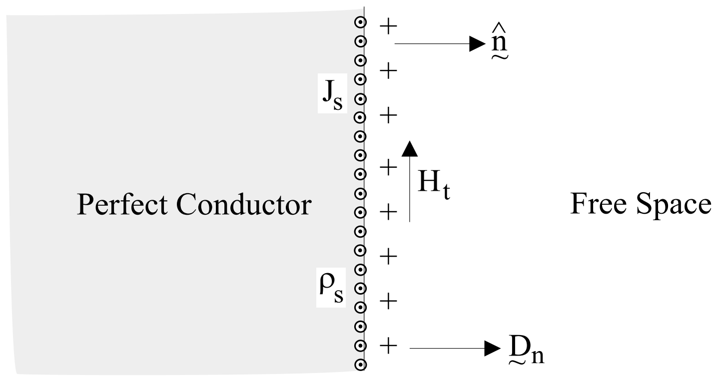


Figure 6.3: Boundary conditions at a perfect conductor surface.

the forms already quoted as equation 6.1 to 6.4 with E_{t1} and D_{n1} set to zero, viz

$$
\begin{aligned}
\mathsf{E}_{t2} &= 0 && (6.13)\\
D_{n2} &= \rho_s && (6.14)\\
\hat{\mathbf{n}} \times (\mathsf{H}_{t2} - \mathsf{H}_{t1}) &= \mathsf{J}_s && (6.15)\\
B_{n2} - B_{n1} &= 0 && (6.16)
\end{aligned}
$$

For time varying fields we will consider in particular the sinusoidal steady state in which we represent vector fields by block capitals as shown in Figure 6.3.

Because in the sinusoidal steady state all fields in the perfect conductor are now zero we will drop the subscript 2 from the fields in the adjacent space.

In the light of the above discussion, and realising that both $\mathbf{J}_s$ and ρ_s are allowed, we now have boundary conditions in the form

$$
\begin{aligned}
\mathbf{E}_t &= 0 && (6.17)\\
\mathrm{D}_n &= \rho_s && (6.18)\\
\hat{\mathbf{n}} \times \mathbf{H}_t &= \mathbf{J}_s && (6.19)\\
\mathrm{B}_n &= 0 && (6.20)
\end{aligned}
$$

We should note the change in notation to that appropriate to the sinusoidal stead state. This final set of boundary conditions is the most commonly encountered in electromagnetic theory.

We note in relation to Figure 6.3 above, the reference directions for $\mathbf{J}_s$ and $\mathbf{H}_t$ are easily established through the right hand rule applied to $\mathbf{J}_s$.

Chapter 7

ELECTROMAGNETIC ENERGY AND FORCES

7.1 Introduction

In this chapter we will consider the storage and transformation of energy in electromagnetic systems.

In the evolution of physical theory, two of the most durable principles have been that of *conservation of charge* and that of *conservation of energy.* Although the latter has in this century been generalised to the concept of conservation of mass-energy, we are not in this course concerned with mass-energy transformations, and our view of the principle will be simply that of conservation of energy.

The principle of conservation of energy, and the fact that electromagnetic fields can exert forces which can do work, inevitably leads to the idea that *energy can be stored in electromagnetic systems.* Pursuing mathematical formulae for the amount of energy stored leads to the discovery that very general expressions for that quantity can be developed in terms of the *field variables*, rather than in terms of the distribution of sources. A simple way of looking at this result is that the energy is stored *within the field itself.* This viewpoint is very commonly adopted in the study of energy and power flow in electromagnetic systems.

7.1.1 Level of treatment

In this treatment of energy storage and power flow, some results will be advanced by inductive generalisation of formulae which can be established readily only in simple geometries. In this aspect the present chapter may appear to differ from earlier chapters, where an attempt was made to produce results by deduction from basic principles. Such a difference, if it does exist, should not be seen as undermining the validity of the results to be presented here. In any scientific work, the acceptance of any theory is only justified by its continuing correctly to predict experimentally verifiable results. The validity of the general formulae to be offerred below is amply justified by their success in providing a framework for the prediction of experimentally verifiable observations.

7.1.2 Methods of energy input

In a general electromagnetic system we can identify the following methods of exchanging energy between that system and its environment.

- Moving charged bodies in an electric field. The forces on the charged bodies result in the performance of mechanical work.
- Injecting currents from voltage sources into terminals of a network. It is usual to regard this as an example of the delivery of electrical power.
- Moving polarised dielectrics in electric fields. The forces and torques on the polarised bodies result in the performance of mechanical work.
- Moving current carrying wires in a magnetic flux density. The forces on the wires require the performance of mechanical work. *It is also true that the movement of the wires may cause changing magnetic fields which induce voltages in the circuits in which the currents flow, causing an exchange of electric energy of a type described earlier.*
- Moving magnetised media in a magnetic field. The forces and torques exerted by the field on the poles of the media involve the performance of mechanical work. *It is also true that changing fluxes caused by the movement may induce voltages in circuits which may be carrying currents to produce the magnetic field in which the motion is taking place.*

7.2 Electromagnetic Forces

In the absence of a detailed treatment here, please refer to the Fields Section of the Level 2 Fields and Energy Conversion notes by the current author.

7.3 Simple Energy Storage Formulae

7.3.1 Linear electrostatic case

In the simple case of a distribution of point charges which are assembled from infinitely separated points into a final position in a linear homogeneous dielectric medium of dielectric permittivity ϵ, it is possible to show from a study of the mechanical work done that the total stored energy, which we define as equal to the mechanical work done, is given by

$$U_e = \frac{1}{2}\,\epsilon \int_v \mathsf{E} \cdot \mathsf{E} \; dv \tag{7.1}$$

where the integral is over the entire region of the field. In the case of linear media which we are considering here one of several possible alternative versions of this formula is clearly

$$U_e = \frac{1}{2} \int_v \mathsf{E} \cdot \mathsf{D} \; dv \tag{7.2}$$

In another simple case when an initially uncharged parallel plate capacitor is charged from a suitably variable voltage source the total stored energy, which we reckon is equal to the electrical energy delivered by the source, is given by the same two formulae quoted above.

In this second example we may see a basis for regarding the *second form* of the above two formulae as being *more closely related to the physical processes.* This is because during the charging process the incremental component of the work done is obviously given by vdq where v is the voltage so far developed across the capacitor and dq is the element of charge being added. In this expression there is a direct relation between v and E, the electric field within the capacitor, and there is also a direct relation between dq and $\mathbf{d}\mathsf{D}$, the change in electric flux density in the capacitor. Thus as far as the *incremental component* of work done in this change, it seems that the most directly appropriate formula is

$$dU_e = \int_v \mathsf{E} \cdot \mathbf{d}\mathsf{D}\, dv \tag{7.3}$$

rather than any alternative formula in which we might substitute for D in terms of $\epsilon\mathsf{E}$ or for E in terms of D/ϵ. This argument, which might appear a little thin in the linear dielectric case, becomes quite compelling in the non-linear dielectric case, where such substitutions are not possible.

7.3.2 Linear magnetostatic case

A not entirely parallel examination of the work done, perhaps by mechanical or perhaps by electrical means, to create a magnetic field leads to the result

$$U_m = \frac{1}{2}\,\mu \int_v \mathsf{H} \cdot \mathsf{H}\, dv \tag{7.4}$$

for the total energy stored. Of course in the case of a linear medium this result has several alternative forms, one of which is

$$U_m = \frac{1}{2} \int_v \mathsf{H} \cdot \mathsf{B}\, dv. \tag{7.5}$$

To see which of these results appears to be the closer to physical processes, we consider the case where the field results from the creation in a toroidal inductor of a current by means of a suitably variable voltage generator connected to the inductor terminals. In any portion of the field creation process, we find the power being delivered to the circuit is the product of the total current i (which is related to the total magnetic field H) and the currently induced voltage v (which is related to the time rate of change of the flux density B).

Thus in an incremental change of flux taking place over time δt, we will find the component $\delta U_m = vi\delta t$ of energy change is related to the product of H, $\partial\mathsf{B}/\partial t$ and δt i.e. to the product of H and $\delta\mathsf{B}$. Thus as far as the *incremental component* of energy supplied to bring about this field change is concerned, it seems that the most directly appropriate formula is

$$dU_m = \int_v \mathsf{H} \cdot \mathbf{d}\mathsf{B}\, dv. \tag{7.6}$$

7.4 General Formulae for Energy Change

In the light of the results discussed in the last two sections we will postulate that the general expression for the change in stored energy in an electromagnetic system in response to changes in fields is

$$dU = \int_v \{\mathsf{E} \cdot \mathbf{d}\mathsf{D} + \mathsf{H} \cdot \mathbf{d}\mathsf{B}\} dv \tag{7.7}$$

This general result is, in simple situations amenable to analysis, in accord with theoretical results based on Maxwell's equations and the force law, and in more complex geometries is in accord with experiments, and will thus be regarded as a correct statement in all cases of the changes in stored energy in an electromagnetic field.

Because it is very convenient to regard the energy as *actually stored within the field*, we will remove the integration and say that at any point the change in energy stored *per unit volume* in a field, when the field changes, is

$$dW = \mathsf{E} \cdot \mathbf{d}\mathsf{D} + \mathsf{H} \cdot \mathbf{d}\mathsf{B} \tag{7.8}$$

where W denotes the *energy stored per unit volume* of space at a point at which the field vectors appearing in the equation above apply.

If we divide by an incremental time dt over which the energy change has taken place and proceed to the limit, recognising the independence of both the space and time variables, we obtain

$$\boxed{\frac{\partial W}{\partial t} = \mathsf{E} \cdot \frac{\partial \mathsf{D}}{\partial t} + \mathsf{H} \cdot \frac{\partial \mathsf{B}}{\partial t}}. \tag{7.9}$$

This is the fundamental equation from which we will derive in the next Section the concept of the Poynting vector.

7.5 Derivation of Poynting Vector

7.5.1 Analysis

Maxwell's equations convert the above equation to

$$\frac{\partial W}{\partial t} = \mathsf{E} \cdot (\mathbf{curl}\ \mathsf{H} - \mathsf{J}\ ) - \mathsf{H} \cdot (\mathbf{curl}\, \mathsf{E} - 0) \tag{7.10}$$

where the absence of free magnetic charges is emphasised by the placement of a zero in the second bracket. Using the vector identity

$$\mathrm{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{curl}\ \mathbf{A} - \mathbf{A} \cdot \mathbf{curl}\ \mathbf{B} \tag{7.11}$$

we are able to manipulate the above equation into the form

$$\frac{\partial W}{\partial t} + \mathsf{E} \cdot \mathsf{J}\ + \mathrm{div}\,(\mathsf{E} \times \mathsf{H}\,) = 0. \tag{7.12}$$

7.5.2 Interpretation

Now the first term represents the rate of increase of stored energy per unit volume. The second term represents the rate of working per unit volume of the field on the conduction currents J . These currents might result in the dissipation of energy in the resistance of the medium, or might, if flowing in a pair of external terminals, result in the supply of electric power to or removal of electric power from the field system. The third term must, by conservation of energy, represent the *rate of flow of energy per unit volume out of the volume by electromagnetic means*. In fact the energy flows out in the form of photons, but we are not working to a microscopic scale, so quantization of the field and such conditions do not concern us. To us it looks as if energy is being continuously transported by the field in an amount div $(\mathsf{E} \times \mathsf{H})$ per unit volume, per unit time.

Now by Gauss' theorem

$$\oint_S (\mathsf{E} \times \mathsf{H}) \cdot \mathbf{ds} = \int_v \mathrm{div}\,(\mathsf{E} \times \mathsf{H})\, dv. \tag{7.13}$$

So we identify $\mathsf{E}\times\mathsf{H}$ as the *power flow per unit area* across any surface.

7.5.3 Real Poynting vector

Because of its importance this vector $\mathsf{E}\times\mathsf{H}$ is called the *Poynting Vector* and is normally in the time domain denoted by N .

$$\boxed{\mathsf{N} = \mathsf{E} \times \mathsf{H}}. \tag{7.14}$$

We note that at this stage the Poynting vector has been defined of terms real vectors E and H; that those fields may be either steady or time varying, and the Poynting vector is in general a time-varying quantity. For steady or time varying fields, the vector N can contain a steady component and a fluctuating component. In a *sinusoidal steady state* situation there are two components, one at d.c. and one at twice the operating frequency.

7.5.4 Complex Poynting vector

Frequently we are interested in *sinusoidal steady state solutions* only, so we define for these the *Complex Poynting Vector*

$$\boxed{\mathbf{S} = \frac{1}{2}\,\mathbf{E} \times \mathbf{H}^*}. \tag{7.15}$$

The reason for the definition of this additional vector will be made clear in the next section.

7.5.5 Interpretation

Let us examine the relation between complex $\mathbf{S}$ and the real N . Since the real electric field is given in terms of the corresponding complex phasor by

$$\mathsf{E}(x,y,z,t) = \frac{\mathbf{E}(x,y,z)e^{j\omega t} + \mathbf{E}^*(x,y,z)e^{-j\omega t}}{2} \tag{7.16}$$

and the real magnetic field is given in terms of the corresponding complex phaser by

$$\mathsf{H}(x,y,z,t) = \frac{\mathbf{H}(x,y,z)e^{j\omega t} + \mathbf{H}^*(x,y,z)e^{-j\omega t}}{2} \tag{7.17}$$

we find that the real Poynting vector can be written in the form

$$\mathsf{E} \times \mathsf{H} = \frac{\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}}{4} + \frac{\mathbf{E} \times \mathbf{H}\, e^{2j\omega t} + \mathbf{E}^* \times \mathbf{H}^*\, e^{-2j\omega t}}{4}$$

If we take the time average of each side, the second term on the right hand side contributes zero and hence

$$< \mathsf{N} > = \frac{\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}}{4} = \frac{\mathbf{S} + \mathbf{S}^*}{2} \tag{7.18}$$

The right hand side is just the real part of $\mathbf{S}$, so we have shown that

$$\boxed{< \mathsf{N} > = \Re e\left\{\mathbf{S}\right\}}. \tag{7.19}$$

Expressing this result in words, we say that the real part of the complex Poynting vector is equal to the time average of the real Poynting vector. The definition in equation 7.15 was chosen so that this relation would result.

Thus the complex Poynting vector provides us with a simple basis for calculating the *time average of the power flow* when we know the electromagnetic fields in terms of their phasors, and we do not wish to know the details of the time variation of those fields.

Chapter 8

INTRODUCTION TO ELECTROMAGNETIC WAVES

8.1 Introduction

Our objective in this section is to derive the properties of simple plane electromagnetic waves in a linear medium free of charges and currents, of which empty space provides an example. We will find that such waves have properties that are easily and fruitfully described without the use of complicated equations.

8.2 Fundamental Equations

8.2.1 Maxwell's equations

We will suppose the medium we are studying is either free space or a *homogeneous lossless linear medium*, i.e. it is characterised by a constant μ and ϵ, and that μ and ϵ are real, and the conductivity σ is zero. We assume also that there are no free charges or currents. Then Maxwell's equations in the frequency domain become

$$\begin{aligned} \mathbf{curl}\,\mathbf{E} &= -j\omega\mu\,\mathbf{H} && (8.1)\\ \mathbf{curl}\,\mathbf{H} &= j\omega\epsilon\,\mathbf{E} && (8.2)\\ \mathrm{div}\,\epsilon\mathbf{E} &= 0 && (8.3)\\ \mathrm{div}\,\mu\mathbf{H} &= 0. && (8.4) \end{aligned}$$

8.2.2 Relevance

In this instance the divergence equations are already implied by the **curl** equations, and we will therefore focus our attention just on the curl equations.

8.2.3 Helmholz equation

If we combine the two curl equations we find

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \omega^2\epsilon\mu\mathbf{E}. \tag{8.5}$$

The vector identity **curl curl** = **grad** div − ∇^2 gives, in view of the fact that div $\mathbf{E} = 0$,

$$\boxed{\nabla^2 \mathbf{E} = -\omega^2 \epsilon \mu \mathbf{E}}. \tag{8.6}$$

This equation is known as the *three-dimensional wave equation* and we will be concerned with various of its solutions. When any solution for $\mathbf{E}$ is known we can calculate the accompanying $\mathbf{H}$ from equation 8.1 above.

8.3 Wave Terminology

Before proceeding with the details of the solution it is appropriate that we become familiar with the terminology used in the description of wave solutions.

8.3.1 Exponential solutions

This being a second order homogeneous differential equation, we will look for waves with an exponential spatial variation, that is for solutions of the type

$$\boxed{\mathbf{E} = \mathbf{E}_0 \, e^{-\boldsymbol{\gamma} \cdot \mathsf{r}}}. \tag{8.7}$$

where $\mathbf{E}_0$ is a complex phasor independent of position, giving the value of the electric field phasor at the origin.

8.3.2 Propagation vector

In the above equation $\boldsymbol{\gamma}$ is a complex vector which we call the *propagation vector.* It can be decomposed into real and imaginary component vectors as

$$\begin{aligned}
\boldsymbol{\gamma} &= \boldsymbol{\alpha} + j\boldsymbol{\beta} && (8.8)\\
\boldsymbol{\alpha} &= \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k} && (8.9)\\
\boldsymbol{\beta} &= \beta_x \mathbf{i} + \beta_y \mathbf{j} + \beta_z \mathbf{k}. && (8.10)
\end{aligned}$$

So we see the spatial variation of $\mathbf{E}$ is as the product of factors

$$e^{-\boldsymbol{\alpha} \cdot \mathsf{r}} \, e^{-j\boldsymbol{\beta} \cdot \mathsf{r}}. \tag{8.11}$$

8.3.3 Plane wave terminology

In the above equation the first factor changes *amplitude*, the second changes *phase.* The directions of maximum rates of change of amplitude and phase are $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively. The plane perpendicular to $\boldsymbol{\beta}$ is a *plane of constant phase*; that such planes exist is why solutions of this type are called *plane waves.* The plane perpendicular to $\boldsymbol{\alpha}$ is a *plane of constant amplitude.* If such variations of amplitude exist, i.e. if $\alpha \neq 0$, the wave is called a *non-uniform plane wave.* If no such variation of amplitude exists, i.e. if $\alpha = 0$, the wave is called a *uniform plane wave.*

8.4 Uniform Plane Wave Solutions

8.4.1 Simplification of Maxwell's equations

When $\alpha = 0$ and the z axis is chosen to lie along the direction of $\boldsymbol{\beta}$, we can assert the equations: $\partial/\partial x = 0$, $\partial/\partial y = 0$, and $\partial/\partial z = -j\beta$. With these restrictions Maxwell's equations in the frequency domain become

$$\begin{aligned} j\beta \mathrm{E}_y &= -\mu j\omega \mathrm{H}_x \\ -j\beta \mathrm{E}_x &= -\mu j\omega \mathrm{H}_y \\ 0 &= -\mu j\omega \mathrm{H}_z \end{aligned} \tag{8.12}$$

$$-j\beta \mathrm{E}_z = 0 \tag{8.13}$$

$$\begin{aligned} j\beta \mathrm{H}_y &= \epsilon j\omega \mathrm{E}_x \\ -j\beta \mathrm{H}_x &= \epsilon j\omega \mathrm{E}_y \\ 0 &= \epsilon j\omega \mathrm{E}_z \end{aligned} \tag{8.14}$$

$$-j\beta \mathrm{H}_z = 0. \tag{8.15}$$

8.4.2 Transverse electromagnetic wave solutions

We have above eight equations; four of them require and are satisfied by setting both longitudinal components to zero, i.e. $\mathrm{E}_z = 0$ and $\mathrm{H}_z = 0$. Because of these conditions the resulting waves are called Transverse Electromagnetic (abbreviated as TEM) waves. The remaining four equations, when the j factors are dropped, can be grouped as the two pairs shown below.

$$\begin{aligned} \beta \mathrm{E}_y &= -\mu\omega \mathrm{H}_x \\ \beta \mathrm{H}_x &= -\epsilon\omega \mathrm{E}_y \end{aligned} \tag{8.16}$$

and

$$\begin{aligned} \beta \mathrm{E}_x &= \mu\omega \mathrm{H}_y \\ \beta \mathrm{H}_y &= \epsilon\omega \mathrm{E}_x \end{aligned} \tag{8.17}$$

The pairs of equations above represent two independent solutions with spatial arrangements of **E**, **H** and $\boldsymbol{\beta}$ as illustrated in Figure 8.1.

We note that each of these solutions has the property that the electric field, magnetic field and propagation vector are *mutually orthogonal* and form a *right hand system* when taken in that order. Both have the same velocity as derived below.

Figure 8.1: Mutually orthogonal **E**, **H** and β.

We have one solution as illustrated for each pair of equations. In each case the equations when combined lead to the result

$$\beta^2 = \mu\epsilon\omega^2. \tag{8.18}$$

The wave velocity $c = \omega/\beta$ is then given by

$$\boxed{c = \frac{1}{\sqrt{\mu\epsilon}}}. \tag{8.19}$$

8.4.3 Detailed expression of solutions

In terms of the physically meaningful *time dependent* variables the detailed solutions are therefore

$$\begin{aligned} \mathrm{E}_y &= \Re\left\{\mathrm{E}_1(0)\, e^{j(\omega t-\beta z)}\right\} \\ \mathrm{H}_x &= \Re\left\{\text{-}\mathrm{H}_1(0)\, e^{j(\omega t-\beta z)}\right\} \end{aligned} \tag{8.20}$$

and

$$\begin{aligned} \mathrm{E}_x &= \Re\left\{\mathrm{E}_2(0)\, e^{j(\omega t-\beta z)}\right\} \\ \mathrm{H}_y &= \Re\left\{\mathrm{H}_2(0)\, e^{j(\omega t-\beta z)}\right\} \end{aligned} \tag{8.21}$$

where the amplitudes $\mathrm{E}_1(0)$, $\mathrm{H}_1(0)$ and $\mathrm{E}_2(0)$, $\mathrm{H}_2(0)$ of the wave for two solutions are related by

$$\frac{\mathrm{E}_1(z)}{\mathrm{H}_1(z)} = \frac{\mathrm{E}_1(0)}{\mathrm{H}_1(0)} = \frac{\mu\omega}{\beta} = \frac{\beta}{\epsilon\omega} = \sqrt{\frac{\mu}{\epsilon}} \tag{8.22}$$

$$\frac{\mathrm{E}_2(z)}{\mathrm{H}_2(z)} = \frac{\mathrm{E}_2(0)}{\mathrm{H}_2(0)} = \frac{\mu\omega}{\beta} = \frac{\beta}{\epsilon\omega} = \sqrt{\frac{\mu}{\epsilon}} \tag{8.23}$$

8.4.4 Characteristic impedance of medium

The ratio of the electric field and the magnetic field phasors calculated at any point of space has a constant value called the *wave impedance* of the medium and denoted by η. Thus

$$\boxed{\frac{\mathrm{E}_0}{\mathrm{H}_0} = \eta} \tag{8.24}$$

where

$$\boxed{\eta = \sqrt{\frac{\mu}{\epsilon}}}. \tag{8.25}$$

The value of this important parameter is approximately 120π ohm.

8.4.5 Remarks on polarization

The *polarisation* of the wave is *by convention* defined by the motion of E. Both of the two independent solutions derived above are examples of *linear polarization*, one along the x axis and one along the y axis. In each case E, H, and $\boldsymbol{\beta}$ form a *right hand set* of *mutually orthogonal* vectors.

The general solution for a uniform plane wave can be obtained by superposing any combination, with arbitrary relative phases and amplitudes, of these two solutions. The result is that E then describes an *elliptical path* with arbitrary orientation and eccentricity. Special cases include those of *linear* or *circular* polarization. In all cases the H vector describes a path of the same shape, rotated 90° about the z axis in the right hand sense. The ratio of the peak values of the electric field and magnetic field is a constant equal to the characteristic impedance η defined above.

We note that either sense (i.e. positive or negative) of circular polarization can be synthesised by this procedure.

8.5 Power Flow in Uniform Plane Waves

8.5.1 Calculation

For a uniform plane wave in the forward direction the complex Poynting vector of Section 7.5.4 has the value:

$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ \mathrm{E}_x\mathrm{H}_y^* - \mathrm{E}_y\mathrm{H}_x^* \end{bmatrix} \tag{8.26}$$

.

Since $\mathrm{H}_y = \mathrm{E}_x/\eta$ and $\mathrm{H}_x = -\mathrm{E}_y/\eta$ we may simplify this to

$$\mathbf{S} = \frac{1}{2\eta}\begin{bmatrix} 0 \\ 0 \\ \mathrm{E}_x\mathrm{E}_x^* + \mathrm{E}_y\mathrm{E}_y^* \end{bmatrix} \tag{8.27}$$

.

It is easy to see that the real part of **S** is the same vector.

8.5.2 Interpretation

The form of the above expression shows that the power flow is positive in the $+z$ direction, and that the two components of the electric field *do not interact*.

A similar analysis of power flow in the case of two waves propagating in opposite directions leads to the similar and interesting conclusion that these two waves do not interact in their power flow either. The particular result we can prove is that the net power carried to the right is equal to the power which the rightward propagating wave (if it alone were present) would carry, less the power which the leftward propagating wave would carry to the left if it alone were present.

This result demonstrates a property known as the *power orthogonality of oppositely directed uniform plane waves.*

Because power is a quantity *quadratically* dependent on the field amplitudes, we could not have obtained this last result by superposition. It appears to be an interesting property of plane wave solutions of Maxwell's equations.

8.6 Reflection and Transmission in Lossless Media

We return to lossless media, i.e. put $\sigma = 0$, and we study the problem of reflection and transmission of electromagnetic waves when a single linearly polarized wave is incident on the boundary from the left as illustrated in Figure 8.2.

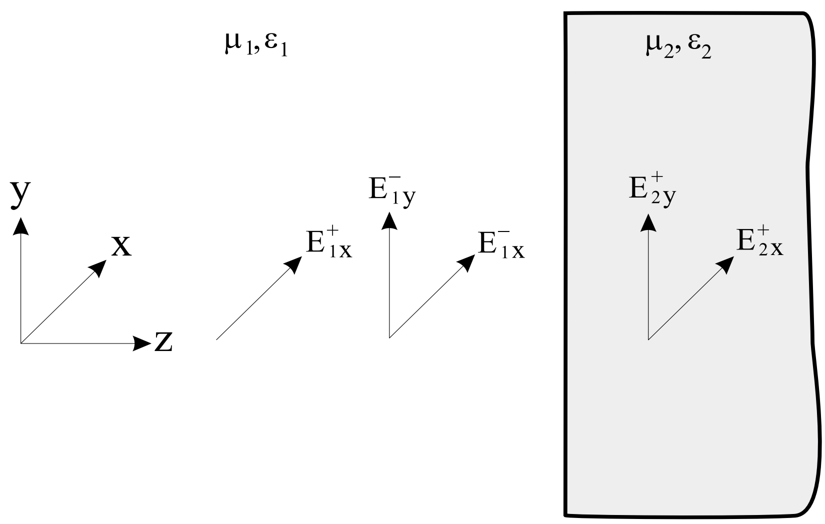


Figure 8.2: Incident, reflected and transmitted waves.

We look for a solution with *incident*, *reflected* and *transmitted* waves, with arbitrary polarization as illustrated. We can write down the solutions for the incident, reflected and transmitted wave components immediately from a knowledge of η and β and the general properties of TEM waves.

$$
\begin{aligned}
\beta_1 &= \omega\sqrt{\mu_1\epsilon_1} \\
\text{and} \qquad \beta_2 &= \omega\sqrt{\mu_2\epsilon_2}
\end{aligned}
\tag{8.28}
$$

$$
\begin{aligned}
\eta_1 &= \sqrt{\frac{\mu_1}{\epsilon_1}} \\
\text{and} \qquad \eta_2 &= \sqrt{\frac{\mu_2}{\epsilon_2}}.
\end{aligned}
\tag{8.29}
$$

The results we write below are obtained by noting that $|\mathrm{E}/\mathrm{H}| = \eta$ and that $\mathbf{E}$, $\mathbf{H}$ and $\boldsymbol{\beta}$ form a right-hand system.

We begin by introducing in the arrays below appropriate notation for the electric field components of the incident, reflected and transmitted waves.

Incident Reflected Transmitted

$$
\mathbf{E}_1^+ = \begin{bmatrix} \mathrm{E}_{1\mathrm{x}}^+ \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{E}_1^- = \begin{bmatrix} \mathrm{E}_{1x}^- \\ \mathrm{E}_{1y}^- \\ 0 \end{bmatrix} \qquad \mathbf{E}_2^+ = \begin{bmatrix} \mathrm{E}_{2x}^+ \\ \mathrm{E}_{2y}^+ \\ 0 \end{bmatrix} \tag{8.30}
$$

Next we use the relations between electric and magetic field components for plane waves to write the magnetic field components of the incident, reflected and transmitted waves in terms of the just introduced electric field components. The results are

Incident Reflected Transmitted

$$
\mathbf{H}_1^+ = \begin{bmatrix} 0 \\ \mathrm{E}_{1x}^+/\eta_1 \\ 0 \end{bmatrix} \quad \mathbf{H}_1^- = \begin{bmatrix} \mathrm{E}_{1y}^-/\eta_1 \\ -\mathrm{E}_{1x}^-/\eta_1 \\ 0 \end{bmatrix} \quad \mathbf{H}_2^+ = \begin{bmatrix} -\mathrm{E}_{2y}^+/\eta_2 \\ \mathrm{E}_{2x}^+/\eta_2 \\ 0 \end{bmatrix} \tag{8.31}
$$

Next we apply the *boundary conditions* for the tangential components of both the electric field and the magnetic field across the boundary. The boundary conditions are that the tangential components (the x and y components in this case) of the *electric field* are continuous, (this is aways true, whatever the form of the boundary) and that the tangential components (again the x and y components) of the *magnetic field* are also continuous (this is true for boundaries on which there is no surface current density, as we have here, because on neither side of the boundary is there a perfect conductor).

Applying these boundary conditions gives a total of four simultaneous equations in the variables defined originally in equations 8.30 above. You should as an *exercise*, assemble these equations, and confirm the conclusions outlined below.

The first set of conclusions is that $E_{1y}^- = 0$ and $E_{2y}^+ = 0$, i.e. the transmitted and reflected waves are also *linearly polarized* and are polarised in the same direction as the incident wave. Two of the magnetic field components are also found to be zero.

The second set of conclusions relates to the relative amplitudes of the incident, reflected and transmitted waves. For transmitted and reflected waves we define an *amplitude transmission coefficient* τ and an *amplitude reflection coefficient* ρ given by the equations

$$\tau(0) = \frac{E_{2x}^{+}}{E_{1x}^{+}} = \frac{2(\frac{\eta_2}{\eta_1})}{(\frac{\eta_2}{\eta_1}) + 1} \tag{8.32}$$

$$\rho(0) = \frac{E_{1x}^{-}}{E_{1x}^{+}} = \frac{(\frac{\eta_2}{\eta_1}) - 1}{(\frac{\eta_2}{\eta_1}) + 1} \tag{8.33}$$

where the field quantities to be used are those at the boundary $z = 0$.

Please note that it is *not* true that $\rho + \tau = 1$. (If you are tempted to entertain this belief, it is worth reflecting upon where such temptation comes from, and whether the context in which a previously encountered similar relation is valid here).

What *is* true, and can easily be seen from equations 8.32 and 8.33 to be so, is that

$$\tau = 1 + \rho \tag{8.34}$$

It is useful to recognise that this equation is not only a valid deduction from equations 8.32 and 8.33, but is also a re-statement of the principle that the tangential component of electric field is continuous across the boundary, which is in fact the principle from which equations 8.32 and 8.33 were derived.

The situation we have is completely analogous to the *transmission line problem*, if we regard the electric field as analogous to voltage and magnetic field as analogous to current. To show the connection we define a concept called *wave impedance looking in the $+z$ direction* as

$$Z(z) = \frac{E_x^{total}}{H_y^{total}} \tag{8.35}$$

For a single travelling plane wave $Z(z) = \eta$ and is independent of z. When both waves are present the result is more complicated, and we come to it in a moment. We also define a reflection coefficient looking in the $+z$ direction

$$\rho(z) = \frac{E_{1x}^{-} e^{j\beta z}}{E_{1x}^{+} e^{-j\beta z}} \tag{8.36}$$

This definition is consistent with equation 8.33 above. We can see immediately that

$$\boxed{\rho(z) = \rho(0) e^{2j\beta z}} \tag{8.37}$$

This relation is of the *same form as the corresponding relation obtained with the distributed circuit approach to transmission lines.* The wave impedance at any point is given in equation 8.35 and is in more detail

$$Z(z) = \frac{E_{1x}^{+} e^{-j\beta z} + E_{1x}^{-} e^{j\beta z}}{\left(E_{1x}^{+} e^{-j\beta z} - E_{1x}^{-} e^{j\beta z}\right)/\eta_1} \tag{8.38}$$

With the aid of equation 8.36 this expression can be simplified to

$$\boxed{\frac{Z(z)}{\eta_1} = \frac{1 + \rho(z)}{1 - \rho(z)}} \tag{8.39}$$

We also define a wave impedance $z_n(z)$ normalized with respect to the characteristics of medium 1 and write

$$\boxed{z_n(z) = \frac{Z(z)}{\eta_1} = \frac{1+\rho(z)}{1-\rho(z)}} \tag{8.40}$$

This equation is again *of the same form as the corresponding relation obtained with the distributed circuit approach to transmission lines.* Thus for the situation of a uniform plane wave incident at the normal incidence on the plane boundaries, equations 8.37 and 8.40 are completely equivalent to the corresponding equations for transmission lines terminated in resistive loads (or other transmission lines), and the complete set of Smith Chart procedures for studying transmission and reflection become available. Several exercises will be based on this property.

The *polarization* of the reflected and transmitted waves is determined by that of the incident wave. We have studied here a particular linear polarization, but this polarization could have been in any direction. All the results we have here for Z, ρ, τ in terms of η_1 and η_2 would be the same. One can form also by superposition of linear polarizations of suitable phases, amplitudes and directions, any arbitrary elliptical polarization for which all these relations remain valid.

But an interesting thing happens to the polarization of the reflected wave. The *shape* and *orientation* of the polarisation ellipses of the reflected wave and the transmitted wave are both the same as that of the incident wave, but for the reflected wave the *sense of polarization is reversed.*

8.7 Reflection From Perfect Conductors

We want to find the expression for the electromagnetic field components of the reflected wave when a uniform plane wave is incident upon a metal boundary. We plan to use our knowledge of the characteristic of TEM waves to guide ourselves to a solution rather than getting lost in much algebra. We outline below the series of steps we follow in constructing the solution, in the hope that this outline will serve as a guide to solving similar problems.

1. *Decide on suitable co-ordinates*, more to fit the boundary than to fit the wave. The reason for doing this should be discussed. The co-ordinae system is illustrated in Figure 8.3

2. *Consider independently two linear polarizations*, once again chosen to fit the boundary. We choose here the x polarisation first.

3. *Set up an incident TEM plane wave.* Using complex notation and the superscript i to indicate the incident wave, the wave is described by the equations

$$\mathbf{E}^i = \begin{bmatrix} \mathrm{E}_0 \\ 0 \\ 0 \end{bmatrix} e^{-j\boldsymbol{\beta}^{\mathrm{i}}\cdot\mathrm{r}} \tag{8.41}$$

where

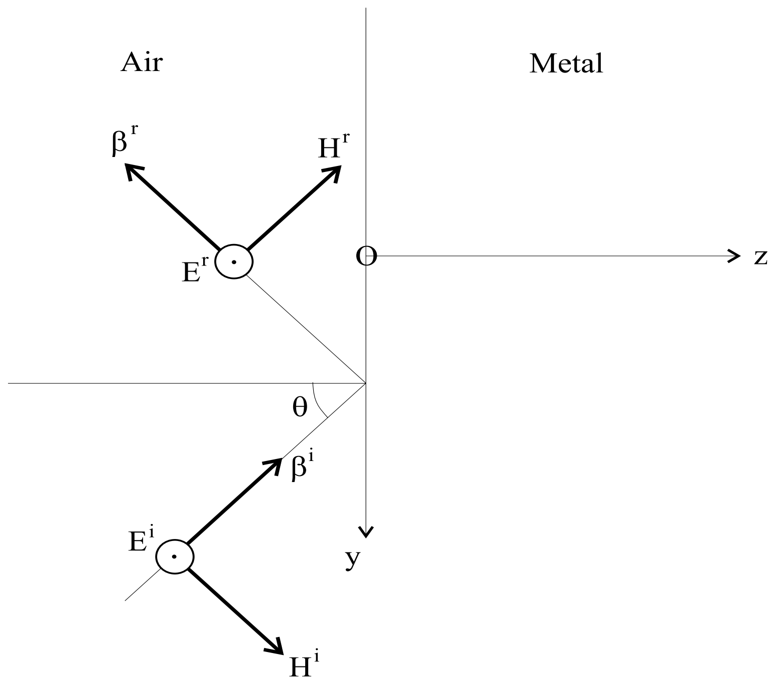


Figure 8.3: Illustration of incident and reflected waves.

$$\boldsymbol{\beta}^i = \begin{bmatrix} 0 \\ -\beta \sin\theta \\ \beta\cos\theta \end{bmatrix} \tag{8.42}$$

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{8.43}$$

Therefore

$$\mathbf{E}^i = \begin{bmatrix} \mathrm{E}_0 \\ 0 \\ 0 \end{bmatrix} e^{j\beta(y\sin\theta - z\cos\theta)} \tag{8.44}$$

We know **H** is in phase with **E**, the magnitude is $|\mathrm{E}/\eta|$ and the direction is such that **E**, **H** and $\boldsymbol{\beta}$ form a right hand system. Hence

$$\mathbf{H}^i = \frac{\mathrm{E}_0}{\eta} \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} e^{j\beta(y\sin\theta - z\cos\theta)} \tag{8.45}$$

4. *Suppose that the reflected wave is plane polarized* and has the angle of reflection equal to the angle of incidence. The plane polarization assumption at least is plausible,

and once made it is clear that the plane must be along x to allow the tangential **E** to be cancelled at the boundary. Conservation of energy requires that the magnitudes of the electric fields must be the same, but there could be a difference in phase. The electric field function which expresses these constraints on the propagation direction and polarisation direction is

$$\mathbf{E}^r = \mathrm{E}_0^r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{j\beta(y\sin\theta + z\cos\theta)} \tag{8.46}$$

from which we may derive

$$\mathbf{H}^r = \frac{\mathrm{E}_0^r}{\eta} \begin{bmatrix} 0 \\ -\cos\theta \\ \sin\theta \end{bmatrix} e^{j\beta(y\sin\theta + z\cos\theta)} \tag{8.47}$$

5. *Note* that since $\mathbf{E}^i$ and $\mathbf{E}^r$ each satisfy Maxwell's equations by the way we have constructed them, so will the total field $\mathbf{E} = \mathbf{E}^i + \mathbf{E}^r$, $\mathbf{H} = \mathbf{H}^i + \mathbf{H}^r$, because Maxwell's equations are linear equations.

6. *Check the boundary conditions.* These are simply $\mathbf{E}_t = 0$. This will ensure that $\mathrm{B}_n = 0$. A surface charge density q_s and current density K will arise on or in the metal, but we don't enquire what they are at this stage. The boundary conditions are satisfied by $\mathrm{E}_0^r = -\mathrm{E}_0$. So the total field is given by

$$\mathbf{E} = \begin{bmatrix} \mathrm{E}_0 \\ 0 \\ 0 \end{bmatrix} e^{j\beta(y\sin\theta - z\cos\theta)} + \begin{bmatrix} -\mathrm{E}_0 \\ 0 \\ 0 \end{bmatrix} e^{j\beta(y\sin\theta + z\cos\theta)} \tag{8.48}$$

Note that *at every point* in the plane $z = 0$, the tangential electric field has vanished. This is an important requirement of the boundary conditions. The total magnetic field is then

$$\mathbf{H} = \frac{\mathrm{E}_0}{\eta}\left(\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} e^{j\beta(y\sin\theta - z\cos\theta)} + \begin{bmatrix} 0 \\ \cos\theta \\ -\sin\theta \end{bmatrix} e^{j\beta(y\sin\theta + z\cos\theta)} \right) \tag{8.49}$$

We note that in the plane $z = 0$ the normal component of **H** is zero, as it must be, so that the normal component of **E** will be zero.

7. *Conclusion.* Since the total field has satisfied Maxwell's equations and the boundary conditions we have the complete solution to the reflection problem.

8.8 Exercises on Metallic Reflection

1. Follow the procedure above for the case when the incident wave is linearly polarized along the direction occupied by **H** in the example just given, i.e. in the direction

$$\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$$

2. For the two different polarizations of incident wave, investigate the nature of the surface currents which flow in the plane $z = 0$.

Chapter 9

PLANE WAVES IN DISSIPATIVE MEDIA

9.1 Sources of Loss

In a disipative medium the sources of loss we may encounter are *dielectric loss*, *magnetic hysteresis loss*, and *conduction loss*. We present here a treatment of the mechanism of conduction loss; the other forms of loss may be treated in a manner which leads to similar results macroscopically, even if the microscopic interactions are suppressed. A brief discussion of the mechanism of loss on an atomic scale is provided in Reference 1, which may be consulted by those who are interested.

We will work in the below analysis in the frequency domain.

9.2 Maxwell's Equations in Conducting Media

We quote first the *constituent relations* to see whether it is appropriate to use **E** and **H** or whether we need **D** and **B** as well. These are

$$\boxed{\begin{array}{r}\mathbf{D} = \epsilon\mathbf{E}\\ \mathbf{B} = \mu\mathbf{H}\\ \mathbf{J} = \sigma\mathbf{E}\end{array}} \tag{9.1}$$

It is appropriate to continue to use **E** and **H** so *Maxwell's equations* are

$$\boxed{\begin{array}{rcl}\nabla\times\mathbf{E} &=& -j\omega\left(\mu\mathbf{H}\right)\\ \nabla\times\mathbf{H} &=& j(\omega\epsilon - j\sigma)\mathbf{E}\\ \nabla\cdot\mathbf{E} &=& \rho/\epsilon\\ \nabla\cdot\mathbf{H} &=& 0\end{array}} \tag{9.2}$$

An argument will be provided in lectures to show that the right hand side of the third equation is zero when $\omega\epsilon$ and σ are both different from zero. This means that although currents will flow, they will do so without having a divergence which can create a time varying charge density. Thus Maxwell's equation's become

$$\boxed{\begin{array}{rcl} \nabla \times \mathbf{E} & = & -j\omega\,(\mu\mathbf{H}) \\ \nabla \times \mathbf{H} & = & j(\omega\epsilon - j\sigma)\mathbf{E} \\ \nabla \cdot \mathbf{E} & = & 0 \\ \nabla \cdot \mathbf{H} & = & 0 \end{array}} \tag{9.3}$$

We note that these are precisely the equations we encounter in the absence of conductivity, *with the difference that $\omega\epsilon$ is replaced by $\omega\epsilon - j\sigma$.*

9.3 Non-Uniform Plane Wave Solutions

9.3.1 General discussion

We look for *plane wave solutions* of the form $e^{-\boldsymbol{\gamma}\cdot\mathbf{r}}$, in which $\boldsymbol{\gamma}$ is the complex vector

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} + j\boldsymbol{\beta} \tag{9.4}$$

It will be recalled from the previous discussion that there exist planes of constant amplitude ($\perp\ \boldsymbol{\alpha}$) and planes of constant phase ($\perp\ \boldsymbol{\beta}$). It may be shown that in the lossless medium case (i.e. when $\sigma = 0$) that $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 0$. Thus in a lossless medium either there is *no attenuation* or the *attenuation is at right angles to the propagation.* A proof of this result appears in Reference 2.

We did not have occasion to discuss any such waves in the lossless case but would have encountered them if we looked carefully at the waves in medium 2 when *total internal reflection* of electromagnetic energy obliquely incident on a dielectric interface occurs in medium 1.

In the case which we pursue here of a lossy medium, there is no need for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to be at right angles; in general they can be at some other angle.

In many important cases, however, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ happen to be parallel and we look now for solutions of that type. Thus we will look for waves whose *direction of propagation and direction of attenuation are the same.*

9.3.2 TEM wave solutions

We now note again that the equations 9.3 which we seek to solve are the same as those of the lossless case studied in Chaper 8, *except that $\omega\epsilon$ has been replaced by $\omega\epsilon - j\sigma$.* Thus a corresponding set of solutions, with the same replacement, can be derived. We will quote below the results which are obtained by making this replacement.

First the *propagation vector*, in the z direction, is

$$\boxed{\gamma = \alpha + j\beta = j\sqrt{\omega\mu(\omega\epsilon - j\sigma)}}. \tag{9.5}$$

Next the *wave impedance* is

$$\boxed{\eta = \frac{\mathrm{E}_1(0)}{\mathrm{H}_1(0)} = \frac{\mathrm{E}_0}{\mathrm{H}_0} = \sqrt{\frac{\omega\mu}{\omega\epsilon - j\sigma}}}. \tag{9.6}$$

The *spatial relation* of the field components is preserved. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ are still at right angles, and $\mathbf{E}$, $\mathbf{H}$ and $\boldsymbol{\beta}$ still form a right hand system of vectors. Note, however, that **E** and **H** are no longer in time phase, because η is no longer real.

We can distinguish now the important practical cases of *small loss* and *large loss.*

9.3.3 Medium of small loss

We say the medium has *small loss* when $\sigma \ll \omega\epsilon$ i.e. conduction current $\ll$ displacement current. Then the above two equations can be put into the approximate forms

$$\boxed{\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}} \tag{9.7}$$

i.e. half the product of the conductivity and wave impedance for a lossless medium,

$$\boxed{\beta \approx \omega\sqrt{\mu\epsilon}} \tag{9.8}$$

and

$$\boxed{\eta \approx \sqrt{\frac{\mu}{\epsilon}}}. \tag{9.9}$$

The last two equations are the same as for a lossless medium. The approximations being used here are a little less accurate than those quoted in Reference 1, wherein second order corrections to β and η are provided, but the expessions quoted here are well suited to both practical application and the furter theoretical development found in later chapters.

9.3.4 Medium of large loss

We say the medium has *large loss* (for transmission of waves through that medium) when $\sigma \gg \omega\epsilon$, i.e. conduction current $\gg$ displacement current. Then the equations at the end of Section 9.2 can be put in the different approximate form

$$\boxed{\gamma \approx (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}} \tag{9.10}$$

and

$$\boxed{\eta \approx (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}}. \tag{9.11}$$

These approximations may be obtained from the formulae for the propagation constant and wave impedance of a lossless medium by replacement of $j\omega\epsilon$ by σ. Here we see that *displacement current* has been replaced by *conduction current*. In terms of α and β

$$\boxed{\begin{array}{rcl} \alpha & = & \sqrt{\frac{\omega\mu\sigma}{2}} \\ \beta & = & \sqrt{\frac{\omega\mu\sigma}{2}} \end{array}}. \tag{9.12}$$

We see that the rate of phase change (in radians per metre) and attenuation (in nepers per metre) are equal. The distance for an attenuation by a factor e is defined as the *skin depth* δ.

$$\boxed{\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}}. \tag{9.13}$$

It is common to define the *surface resistivity per square* R_s as

$$\boxed{R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \frac{1}{\delta\sigma}} \tag{9.14}$$

in terms of which the wave impedance is

$$\boxed{\eta = (1+j)R_s}. \tag{9.15}$$

We note that R_s is equal to the resistance per square of a thin sheet of material of thickness equal to δ in which the current is uniform. Of course the current here is decidedly non-uniform in both amplitude and phase as the wave proceeds into the material; that is why $\eta \neq R_s$. But this remark makes it easy to remember R_s when we know δ.

9.4 Reflection From a Good Conductor

We now consider the situation illustrated in Figure 9.1 in which a plane electromagnetic wave is incident form the left on the boundary at the plane $z = 0$ of well-conducting (but not perfectly conducting) medium.

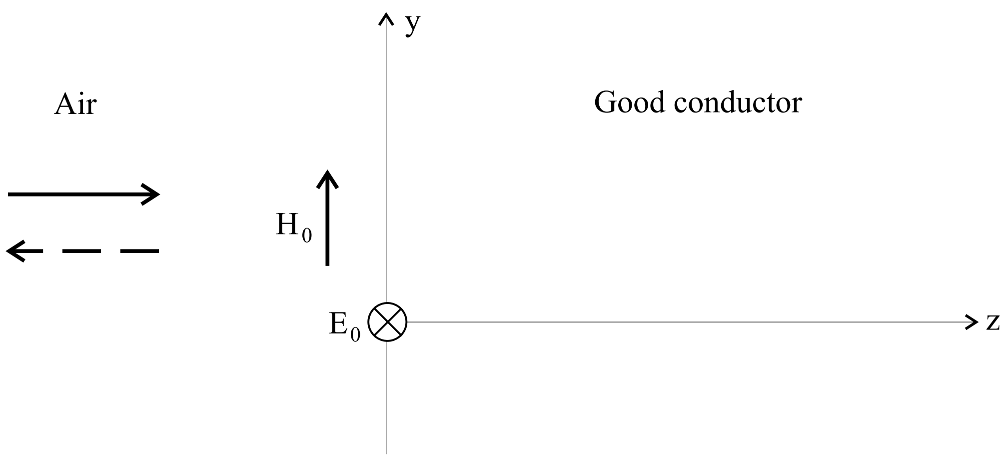


Figure 9.1: Reflection at normal incidence from a good conductor.

We suppose here an incident wave at normal incidence on the boundary of the well-conducting (but not perfectly conducting) medium suffers almost complete reflection. If the reflection were complete the electric field E_0 at the surface would be zero and the total magnetic field H_0 at the surface would be double that of the incident wave.

9.4.1 The surface field

Since the reflection is not quite complete, there will be a *small tangential component of electric field* at the surface. We cannot calculate it yet, but we expect that it will be small.

The magnetic field will, however, still be almost double the magnetic field corresponding to the incident wave, and we will find that this fact gives us a basis for calaculation of all the fields if they are considered in an appropriate order.

In studying the penetration of the fields into the conductor then, we suppose the tangential component H_0 of the magnetic field at the surface is given (just double the magnetic field of the incident wave) and is along O_y, and we calculate the remainder of the field quantities from that parameter.

First we calculate the above-mentioned *small electric field* E_0 at the surface, by using the wave impedance

$$\mathrm{E}_0 = (1+j)R_s\mathrm{H}_0 \tag{9.16}$$

Note E_0 is spatially perpendicular to H_0, i.e. it will be along the O_x. Note the 45° *phase factor* as well.

9.4.2 The interior field

Inside the material all the fields propagate as the attenuated plane wave

$$\mathrm{E}_x(x,y,z) = \mathrm{E}_0^{-\alpha z} e^{-j\beta z} = \mathrm{E}_0 e^{-(1+j)z/\delta}$$

Thus in terms of the original tangential magnetic field

$$\mathrm{E}_x(x,y,z) = (1+j)R_s\mathrm{H}_0 e^{-(1+j)z/\delta} \tag{9.17}$$

The volume current density, also directed along O_x is

$$\mathrm{J}_x(x,y,z) = \sigma\mathrm{E}_x = \left(\frac{1+j}{\delta}\right)\mathrm{H}_0 e^{-(1+j)z/\delta} \tag{9.18}$$

9.4.3 Total current flow

Now we calculate the *total current which flows across the $x=0$ plane*, per unit length of the y axis. This is

$$\begin{aligned} \mathrm{K} &= \int_0^\infty \mathrm{J}_x dz \\ &= \int_0^\infty \frac{(1+j)}{\delta}\mathrm{H}_0 e^{-(1+j)z/\delta} dz \\ &= \mathrm{H}_0 \end{aligned} \tag{9.19}$$

In the limit when $\delta \to 0$ as $\sigma \to \infty$ this current becomes a *surface current density* spread over a skin of zero thickness, at right angles to H_0 and equal in magnitude to H_0.

This result was encountered before when we studied, from an idealised point of view, boundary conditions in the presence of perfectly conducting media.

9.4.4 Power dissipation per unit area

If we calculate *the time average power dissipated* per unit area of the interface, this is

$$\begin{aligned} W_a &= \int_0^\infty W_v dz \\ &= \int_0^\infty \frac{2|\mathrm{H}_0|^2 e^{-2z/\delta}}{2\sigma\delta^2} dz \\ &= \frac{|H_0|^2}{2\sigma\delta} \\ &= \frac{|K_x|^2}{2\sigma\delta} \end{aligned} \tag{9.20}$$

This is the same result as we would have obtained if the current had been assumed to flow in a uniform sheet of thickness δ and conductivity σ. It is unexpected that this equality occur, but since it has, it provides a simple basis for remembering the formula.

9.5 Generalisation

Although we have studied the case of *normal incidence* only, the general picture we have derived is considered to the applicable to a wide range of practical cases.

Thus whenever we have a metallic boundary supporting a magnetic field, we will always assume the magnetic field is supported by a total surface current density $\mathbf{K}$ equal in magnitude but at right angles in direction to the tangential component $\mathbf{H}_t$ of $\mathbf{H}$, and that the power dissipated per unit area of surface is

$$\boxed{W_a = \frac{1}{2}|\mathrm{H}_t|^2 R_s} \tag{9.21}$$

where

$$\boxed{R_s = \frac{1}{\delta\sigma}} \tag{9.22}$$

and

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}}. \tag{9.23}$$

We will find this ability to calculate the power lost, due to skin effect, per unit surface area of a boundary, useful in the determination of the attenation of waves in both coaxial lines and in hollow waveguides in Chapter 10.

Chapter 10

PROPAGATION IN GUIDING STRUCTURES

10.1 Introduction

The objectives of this chapter are several fold. The overall objective is to bring as promised in Chapter 1 an *electromagnetic theory* approach to the study of the transmission of signals in various guiding structures.

The guiding structures to be studied include some for which the assumptions of distributed circuit theory are satisfied, and are amenable to analysis in that context, and some for which the assumptions of distributed circuit theory are not satisfied, and for which an electromagnetic theory based analysis is necessary.

In both cases we find the electromagnetic fields which carry signals along the structure. Knowledge of those fields will lead to an understanding of why a distributed circuit analysis is or is not possible for each structure.

A large part of the Chapter will be occupied with analysis of the fields in those structures for which a distributed circuit approach is not applicable. We will find that transmission of signals in this situation has new and interesting properties worthy of study.

We will also be interested in the question of the *attenuation* of the waves as they travel along the guiding structures. It is clear from our frequent assertion that the currents on the conductors on the guiding structures flow on or near the surface, and not through the volume of the conductor, that assigning a resistance per unit length to the conductors is not the simple matter which it is with d.c. currents.

It is also clear that the fact that in some of the guiding structures, current and voltage are not useful concepts will introduce its own difficulties in the calculation of attenuation. Fortunately, a method of calculation of attenuation which is based on the concepts of Chapter 9, and which allows all these difficulties to be bypassed, exists. This calculation attenuation using this method will occupy a significant fraction of this chapter.

10.2 Classification of Guiding Structures

As we explore the wave types which are possible in different types of guiding structure, we will find that there is an important distinction between types of structure which is illustrated in Figure 10.1.

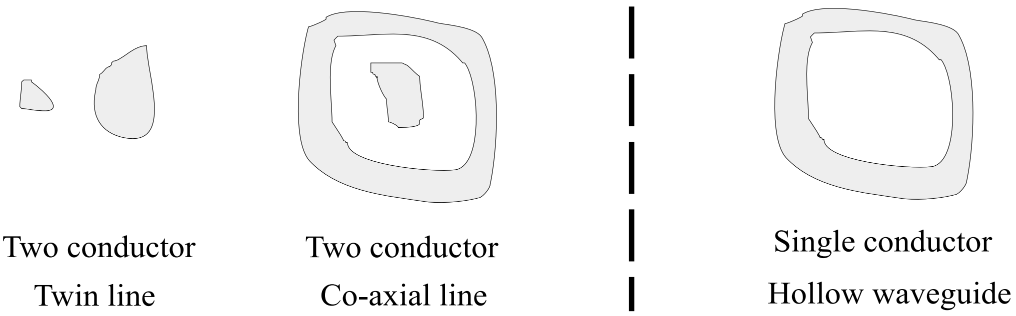


Figure 10.1: Two conductor and single conductor wave guiding structures.

On the left of the dotted line in that figure we see two varieties of two-conductor lines: the *twin line* and the *coaxial line*. These are the types of line which are amenable to analysis using *distributed circuit* concepts. They are, of course, also analysable by *electromagnetic theory* concepts. A really good understanding of the properties of signals on such lines is obtained when *both* approaches to analysis have been used.

On the right of the dotted line is shown a different form of waveguide structure. It may appear incongruous that the waveguide structure has only a single conductor. That conductor is, however, hollow and provides in its interior a space within which electromagnetic waves may propagate. The metal walls of the waveguide assure that the waves do not escape from that environment, thus providing the two important functions of ensuring that the signals reach their intended destination at the load end of the waveguide, and that the signals do not reach any circuits exterior to the waveguide on the way.

We will find that investigation of the propagation of signals in a waveguide will lead us to many interesting concepts such as *propagating modes* and *evanescent modes*, and of *dispersive propagation*, to be developed in Section 10.9.

10.2.1 Assumptions

In our analysis of propagation in guiding structures we will make the following assumptions.

- The analysis will be performed for *sinusoidal signals* described by *complex phasors*.
- The waveguides are assumed to be uniform with respect to the longitudinal axis z.
- Propagation is described by the factor $e^{-\gamma z}$ with $\gamma = \alpha + j\beta$. We are thus allowing for a —textitphase constant β and an *attenuation constant α*.
- To obtain the phase constant β we will assume the waveguide walls have no losses. At that level of analysis the attenuation constant α is zero.
- When the walls have non-zero resistivity we use the lossless analysis to determine an approximate solution for the field configuration, then use that approximae solution to calculate the power lost in the skin currents in the walls, and hence derive a non-zero value for the attenuation constant α. This analysis is philosophically approximate, but practically highly accurate.

10.3 Classification of Wave Types

We will find that different types of wave can propagate in various types of guiding structure illustrated in Figure 10.1. To describe these different wave types we fix our attenuation on the field components in z direction and obtain the classification of wave types shown in Table 10.1.

Abbreviation	Type	Conditions for classification
TEM	Transverse Electromagnetic Waves	$\mathrm{E}_z = 0,\ \mathrm{H}_z = 0$
TE	Transverse Electric Waves	$\mathrm{E}_z = 0,\ \mathrm{H}_z \neq 0$
TM	Transverse Magnetic Waves	$\mathrm{E}_z \neq 0,\ \mathrm{H}_z = 0$

Table 10.1: Classification of wave types.

TEM waves are the simplest and will be dealt with separately. TE and TM waves have many similarities and will be dealt with as far as possible together. These waves may be superposed to form more complicated wave types. Not often is this done. Generally we can arrange so that only one of these simple types does propagate.

10.4 Outline of Analysis

As is usual in electromagnetic field problems our analysis will be based on Maxwell's equations, reproduced below, and the electromagnetic boundary conditions discussed at length in Chapter 6.

10.4.1 Maxwell's equations again

Maxwell's curl equations, written out in full and making use of the assumed form of z axis dependence, are

$$\begin{aligned} \frac{\partial \mathrm{E}_z}{\partial y} + \gamma \mathrm{E}_y &= -j\omega\mu \mathrm{H}_x \\ -\gamma \mathrm{E}_x - \frac{\partial \mathrm{E}_z}{\partial x} &= -j\omega\mu \mathrm{H}_y \\ \frac{\partial \mathrm{E}_y}{\partial x} - \frac{\partial \mathrm{E}_x}{\partial y} &= -j\omega\mu \mathrm{H}_z \end{aligned} \tag{10.1}$$

$$\begin{aligned} \frac{\partial \mathrm{H}_z}{\partial y} + \gamma \mathrm{H}_y &= j\omega\epsilon \mathrm{E}_x \\ -\gamma \mathrm{H}_x - \frac{\partial \mathrm{H}_z}{\partial x} &= j\omega\epsilon \mathrm{E}_y \\ \frac{\partial \mathrm{H}_y}{\partial x} - \frac{\partial \mathrm{H}_x}{\partial y} &= j\omega\epsilon \mathrm{E}_z \end{aligned} \tag{10.2}$$

Instead of writing out the divergence equations we resurrect from the beginning of Chapter 8, two other equations which were obtained by combining the divergence and curl equations. These are

$$\begin{aligned} \nabla^2 \mathbf{E} &= -\left(\frac{\omega}{c}\right)^2 \mathbf{E} \\ \nabla^2 \mathbf{H} &= -\left(\frac{\omega}{c}\right)^2 \mathbf{H}. \end{aligned} \tag{10.3}$$

We develop our argument from the set of equations 10.1 and 10.3.

10.4.2 Transverse field expressions

We derive first a set of equations relating the *transverse field components* to the *longitudinal field components.* By combining various pairs of equations 10.1 and 10.2 in various combinations with various multipliers designed to eliminate selected instances of transverse field components we may show that

$$\boxed{\begin{aligned} \left[\gamma^2 + \left(\tfrac{\omega}{c}\right)^2\right] \mathrm{H}_x &= j\omega\epsilon \frac{\partial \mathrm{E_z}}{\partial y} - \gamma \frac{\partial \mathrm{H_z}}{\partial x} \\ \left[\gamma^2 + \left(\tfrac{\omega}{c}\right)^2\right] \mathrm{H}_y &= -j\omega\epsilon \frac{\partial \mathrm{E_z}}{\partial x} - \gamma \frac{\partial \mathrm{H_z}}{\partial y} \\ \left[\gamma^2 + \left(\tfrac{\omega}{c}\right)^2\right] \mathrm{E}_x &= -\gamma \frac{\partial \mathrm{E_z}}{\partial x} - j\omega\mu \frac{\partial \mathrm{H_z}}{\partial y} \\ \left[\gamma^2 + \left(\tfrac{\omega}{c}\right)^2\right] \mathrm{E}_y &= -\gamma \frac{\partial \mathrm{E_z}}{\partial y} - j\omega\mu \frac{\partial \mathrm{H_z}}{\partial x} \end{aligned}}. \tag{10.4}$$

We see that in these equations we have succeeded in expressing *each of the four transverse field components* in terms of *the two longitudinal field components.* We also rearrange equations 10.3 a little. Because we have already assumed the form of the spatial variation in the z direction, we may put

$$\nabla^2 = \nabla^2 xy + \frac{\partial^2}{\partial z^2} = \nabla^2 xy + \gamma^2. \tag{10.5}$$

The symbol

$$\nabla^2_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{10.6}$$

is known as the two-dimensional Laplacian operator. The equations 10.3 are then

$$\boxed{\begin{aligned} \left[\gamma^2 + \left(\tfrac{\omega}{c}\right)^2\right] \mathbf{E} &= -\nabla^2_{xy}\, \mathbf{E} \\ \left[\gamma^2 + \left(\tfrac{\omega}{c}\right)^2\right] \mathbf{H} &= -\nabla^2_{xy}\, \mathbf{H} \end{aligned}}. \tag{10.7}$$

The quantity in the square brackets in equations 10.4 and equations 10.7 clearly plays an important part in the theory.

The equations 10.4 show that

- If the longitudinal field components are known, the entire field is known, with one proviso; that is $\gamma^2 + (\omega/c)^2 \neq 0$.

- In the case of a TEM wave wherein both longitudinal components of the field are zero, and the entire field will vanish unless $\gamma^2 + (\omega/c)^2 = 0$.

10.5 Transverse Electromagnetic Waves

We now particularise our analysis to the case of purely *transverse electromagnetic* waves. The cases of *transverse electric* waves and *transverse magnetic* waves will be considered in Sections 10.7 and sec:guides:tmm respectively.

10.5.1 Propagation velocity

We have just observed that if a TEM wave does exist in a particular structure its propagation constant and angular frequency must be related by the equation

$$\gamma^2 + \left(\frac{\omega}{c}\right)^2 = 0. \tag{10.8}$$

Neglecting losses this gives the phase constant

$$\beta = \frac{\omega}{c}. \tag{10.9}$$

The waves are thus seen to be *dispersionless* with both phase and group velocities equal to the velocity $c = 1/\sqrt{\mu\epsilon}$. We now proceed to elucidate further properties of TEM waves in guiding structures.

A short discussion in the lecture on the characteristics of *dispersiv*e and *dispersionless* propagation would be appropriate for the benefit of students not familiar with these concepts. Insist upon it!

10.5.2 Electric and magnetic field orthogonality

Note first that from equation 10.1 with E_z and H_z zero we can show that

$$\boxed{\begin{array}{rcl} \mathrm{E}_x & = & \eta\ \mathrm{H}_y \\ \mathrm{E}_y & = & -\eta\ \mathrm{H}_x \end{array}} \tag{10.10}$$

where η has the usual value $\sqrt{(\mu/\epsilon)}$. These equations show that everywhere the transverse components of electric and magnetic fields are *spatially orthogonal*, and that those field compontents are *in time phase*, and their magnitudes *bear a constant ratio η*. The spatial orthogonality of the transverse components of the electric and magnetic fields is similar to that encountered in the study of uniform plane waves in Chapter 8. The results we have obtained here however are more general in that the electric and magnetic fields can vary in both magnitude and direction from point to point, but despite such variation they remain mutually orthogonal, in time phase, and their magnitudes continue to bear a constant ratio.

As noted the electric and magnetic fields are perpendicular to one another, and bear a constant ratio η at any point. In fact $\mathbf{E}$, $\mathbf{H}$, and the propagation vector $\boldsymbol{\beta}$ form a *right handed set* of *mutually orthogonal vectors*, just as they do for TEM waves in free space. However in this case the electric and magnetic field magnitudes can vary from point to point, while maintaining their orthogonality and proportionality. These relationships are clearly shown by the example of the fields, which will be described shortly, for the ordinary transmission line mode in a coaxial line.

10.5.3 Laplaces' equation

Next we note that equations 10.7 become for the TEM mode the pair

$$\begin{aligned} -\nabla^2_{xy}\,\mathbf{E} &= 0 \\ -\nabla^2_{xy}\,\mathbf{H} &= 0 \end{aligned} \tag{10.11}$$

that is, each of the transverse components of the electric and magnetic fields satisfies the two dimensional Laplace equation. Either of these is the equation which must be solved, in conjunction with the boundary conditions, to find the field distribution.

10.5.4 Relation to electrostatic fields

Considering the solution for the electric field, we see that this is the *same equation* as is satisfied by a set of *static electric fields* produced by a set of *stationary charges*. Moreover the *same boundary condition*, viz zero tangential electric field, is encountered in the solution for the electric field components in the electrostatic field problem. As a consequence, many useful solutions for TEM mode field distributions may be derived from problems solved in the *electrostatic context*. We make use of a basic theorem of electrostatics to note that

10.5.5 An important conclusion

There are no TEM modes inside a hollow single-conductor waveguide.

10.6 TEM Modes in Coaxial Cables

The further discussion of TEM modes will be assisted by reference to the important particular case of the *coaxial line* illustrated in Figure 10.2.

10.6.1 Field distribution

Using cylindrical polar co-ordinates r, θ and z, the field components, which may be obtained easily by applying Maxwell's equations in integral form are

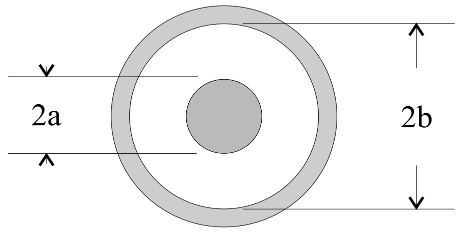


Figure 10.2: Dimensions of coaxial line.

$$\boxed{\begin{aligned} E_r &= E_0 \left(\frac{a}{r}\right) e^{-\gamma z} \\ E_\theta &= 0 \\ E_z &= 0 \end{aligned}} \tag{10.12}$$

$$\boxed{\begin{aligned} H_r &= 0 \\ H_\theta &= H_0 \left(\frac{a}{r}\right) e^{-\gamma z} \\ H_z &= 0 \end{aligned}} \tag{10.13}$$

These results may be obtained readily from the static solutions. We see that **E** and **H** are everywhere perpendicular to one another. The ratio E/H is constant, as expected. It will, by virtue of equation 10.10, be equal to η.

10.6.2 Derivation

The usual way in which these solutions are derived is to assume a field distribution which is plausible in that it contains the radial symmetry suggested by the geometry and satisfies the boundary conditions and incorporates the assumed propagation factor. This leads to the assumption that the electric field is *purely radial* and the magnetic field is *purely circumferential*. The non zero components are then

$$E_r = E_r(r) \tag{10.14}$$

$$H_\theta = H_\theta(r) \tag{10.15}$$

where $E_r(r)$ and $H_\theta(r)$ are functions, yet to be found, of the radial coordinate r.

The application of Gauss' law to the electric field function and Ampere's law to the magnetic field function leads to the conclusion that both have an inverse r variation. The particular forms presented in equations 10.12 and 10.13 have arbitrarily chosen the inner radius a as a *normalising constant*, so that the coefficients E_0 and H_0 can have the units of electric and magnetic fields respectively.

10.6.3 Exercise

It is strongly recommended that students perform in detail the steps outlined in the previous section. It will be found that there are issues not therein described to be considered.

10.6.4 Surface currents

If we calculate the surface current densities K_a and K_b flowing on the surfaces of the inner and outer conductors respectively by applying the familiar boundary conditions we obtain

$$\begin{aligned} \mathrm{K}_a &= \mathrm{H}_0\, e^{-\gamma z} \\ \mathrm{K}_b &= \mathrm{H}_0 \frac{a}{b}\, e^{-\gamma z}. \end{aligned} \tag{10.16}$$

10.6.5 Total current

The total currents flowing along the inner and outer conductors are then

$$\begin{aligned} \mathrm{I}_a &= 2\pi a\ \mathrm{K}_a = 2\pi\ \mathrm{H}_0 a e^{-\gamma z} \\ \mathrm{I}_b &= 2\pi b\ \mathrm{K}_b = -2\pi\ \mathrm{H}_0 a e^{-\gamma z}. \end{aligned} \tag{10.17}$$

Since these have, apart from direction, the *same value* in the two conductors, we do not have trouble in finding an unambiguous definition for the concept of the *current carried by the line*. Such lack of ambiguity is not found with the TE and TM modes to be studied in Sections 10.7 and sec:guides:tmm respectively.

10.6.6 Voltage between conductors

Investigation using the formula 10.12 will show that the line integral of $\mathbf{E}$ from one point on the inner conductor to some point *in the same plane* on the outer conductor is always

$$\mathrm{V} = \mathrm{E}_0 a \log\left(\frac{b}{a}\right) e^{-\gamma z} \tag{10.18}$$

no matter what the path. So we see that there is no difficulty in finding an unambiguous definition for the *potential difference* between the conductors. Such lack of ambiguity is not found with the TE and TM modes to be studied in Sections 10.7 and sec:guides:tmm respectively.

10.6.7 Characteristic impedance

The *impedance of the line* may then be found to be

$$Z = \frac{\mathrm{V}}{\mathrm{I}} = \frac{\mathrm{E}_0 a \log\left(\frac{b}{a}\right) e^{-\gamma z}}{2\pi\ \mathrm{H}_0 a e^{-\gamma z}}$$

Thus

$$Z = \frac{\eta}{2\pi} \log\left(\frac{b}{a}\right) \tag{10.19}$$

which is in agreement with previous formulae derived in Chapter 2 using a distributed inductance and capacitance approach. In this calculation we have, by considering only a forward travelling wave, implicitly assumed a *matched line*, so the value of Z calculated here is the *characteristic impedance* Z_0.

10.6.8 Power flow

The *complex Poynting vector* for the above fields is

$$\begin{aligned} \mathrm{S}_z &= \frac{1}{2}\ \mathrm{E}_r\ \mathrm{H}_\theta^* \\ &= \frac{1}{2}\ \mathrm{E}_0\ \mathrm{H}_0^* \left(\frac{a}{r}\right)^2 \end{aligned} \tag{10.20}$$

and is real. Integration over the element of area $rdrd\theta$ gives the *total power flow*

$$\mathrm{W}_T = \pi\ \mathrm{E}_0\ \mathrm{H}_0^* a^2 \log\left(\frac{b}{a}\right). \tag{10.21}$$

Using the previously derived formulae for voltage and current we see that the result is simply

$$\mathrm{W}_T = \frac{1}{2}\ \mathrm{V}\ \mathrm{I}^* \tag{10.22}$$

as expected.

10.6.9 Attenuation

The *attentuation of the line* is worked out using the physical approximation described at the end of Chapter 9. If the conductor conductivity is σ, the skin depth and surface resistivity are

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \tag{10.23}$$

$$R_s = \frac{1}{(\delta\sigma)} \tag{10.24}$$

We could define a *resistance per unit length* for each conductor and work out the power lost from the total current, but such a procedure does not work for hollow waveguides, which we will study later in this Chapter. Instead we work from the surface current density which is equal to the tangential magnetic field. In terms of the magnetic field the time average power dissipated per unit area of the inner and outer conductors is $\frac{1}{2}R_s\ \mathrm{H}_t\ \mathrm{H}_t^*$. The *power lost per unit length of transmission line* is

$$W_L = \int \frac{1}{2} R_s\ \mathrm{H}_t\ \mathrm{H}_t^* dl \tag{10.25}$$

where the integration is over the *entire boundary* which includes the outer periphery of the inner conductor and the inner periphery of the outer conductor. For the TEM mode in the coaxial line the contributions from the inner and outer conductors are respectively

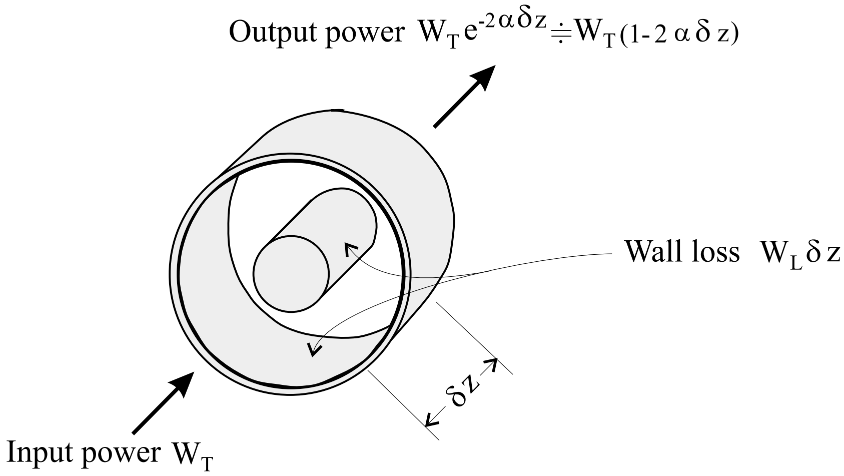
$$\frac{R_s \mid \mathrm{H}_0|^2\, a^2 2\pi}{2b}$$

and

$$\frac{1}{2} R_s \mid \mathrm{H}_0|^2\, 2\pi a.$$

Hence the total time average power dissipation per unit length of the line is

$$W_L = \frac{\pi R_s |\,\mathrm{H}_0|^2 a(a+b)}{b} \tag{10.26}$$



A short length δz of a co-axial line

Figure 10.3: Co-axial line section with wall loss.

By considering the power flows illustrated in Figure 10.3 it may be seen that the *attenuation factor per unit length* α, is related to W_T and W_L by

$$\boxed{\alpha = \frac{W_L}{2W_T}}. \tag{10.27}$$

Students should satisfy themselves of the validity of this result. For the TEM mode in a coaxial line we may use the results obtained in equations 10.21 and 10.26 to obtain

$$\alpha = \frac{R_s(a+b)}{2\eta ab \log\left(\frac{b}{a}\right)} \tag{10.28}$$

The above results, and related quantites for the coaxial line and several other important two-conductor structures, are quoted in Reference 1.

This concludes our discussion of TEM modes. We now turn to a discussion of purely TE modes.

10.7 Transverse Electric Modes

10.7.1 Defining property

The basic equations defining the TE modes are

$$
\begin{aligned}
&& \mathrm{H}_z &\neq 0 \\
\text{and} && \mathrm{E}_z &= 0
\end{aligned}
\tag{10.29}
$$

The extent to which these modes can occur in each of the basic waveguide topologies first illustrated in Figure 10.1 and illustrated again in Figure 10.4, and the practical effect of their occurrence, is shown in Table 10.2.

Of the statements found in the table, we have so far established only that there are no TEM modes in hollow waveguide. The truth of the other table entries will become clear in the following sections.

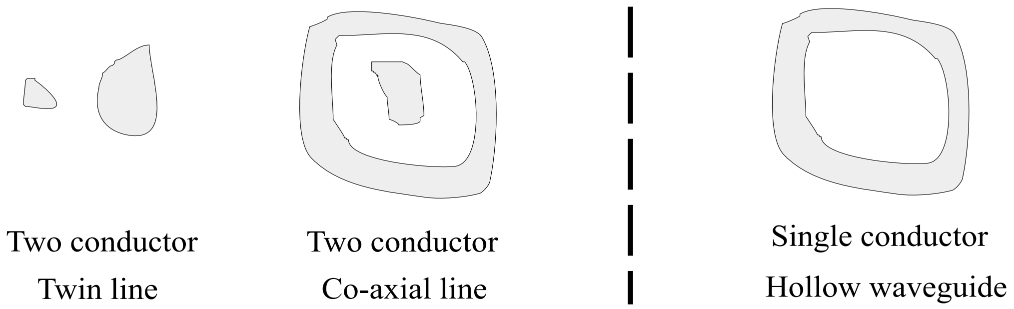


Figure 10.4: Two-conductor and single conductor wave guiding structures.

Two conductor systems			Waveguides		
TEM	yes	useful	TEM	no	
TE	yes	unwelcome	TE	yes	useful
TM	yes	unwelcome	TM	yes	useful

Table 10.2: Possibility and desirability of modes in various propagating structures.

10.7.2 Differential equation

We look for solutions inside hollow pipes because of the importance of such modes. For TE Modes, we have shown in Section 10.4.2 that as the longitudinal field E_z is zero, all transverse fields can be derived from the longitudinal field H_z via equations 10.4. We have also earlier shown that the equation satisfied by H_z is

$$\nabla^2_{xy}\ \mathrm{H}_z = -\left[\gamma^2 + \left(\frac{\omega}{c}\right)^2\right]\ \mathrm{H}_z \tag{10.30}$$

which is a direct quotation from equation 10.7.

10.7.3 Boundary conditions

We now show that the boundary conditions are all contained in the single equation

$$\frac{\partial \mathrm{H}_z}{\partial n} = 0 \tag{10.31}$$

which must apply everywhere on the boundary.

This is an important result as it is expressed in terms of a single variable, and it is the *same variable* for which we have the differential equation 10.30. To show the result we consider an arbitrary point on the boundary. As we have not yet defined particular directions for our x and y axes, we are free to position them as shown in Figure 10.5; in particular we make the direction O_x of the x axis *perpendicular to the boundary at the point of interest.*

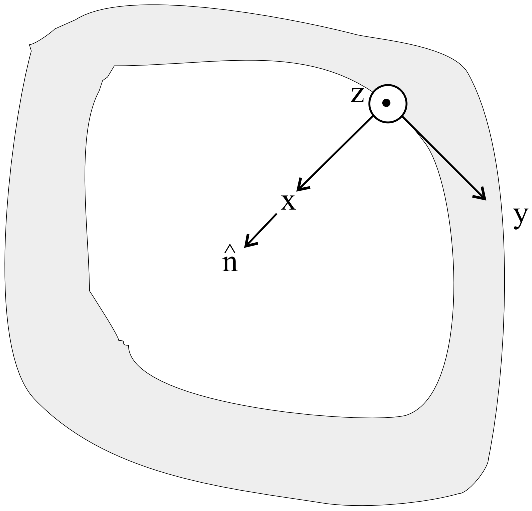


Figure 10.5: Co-ordinates for expression of boundary conditions.

Now the boundary conditions as we know them require that tangential **E** and normal **H** must be zero. This means that

$$\begin{array}{lll} \mathrm{E}_x & & \text{no constraint made} \\ \mathrm{E}_y & = & 0 \text{ (a significant statement)} \\ \mathrm{E}_z & = & 0 \text{ (not a new result)} \end{array} \tag{10.32}$$

$$\begin{array}{lll} \mathrm{H}_x & = & 0 \text{ (a significant statement)} \\ \mathrm{H}_y & & \text{no constraint made} \\ \mathrm{H}_z & & \text{no constraint made} \end{array} \tag{10.33}$$

We note above two significant significant statements made by the boundary conditions. We note also that the purported boundary condition equation 10.31 becomes in the present co-ordinate system

$$\frac{\partial \mathrm{H}_z}{\partial x} = 0 \tag{10.34}$$

and we have of course from the defining equation for TE modes the equation

$$\mathrm{E}_z = 0. \tag{10.35}$$

It may be taken as an *exercise* to show from equations 10.4 that the set of equations

$$\begin{aligned} \frac{\partial \mathrm{H}_z}{\partial x} &= 0 \\ \text{and} \qquad \mathrm{E}_z &= 0 \end{aligned} \tag{10.36}$$

just mentioned and involving the purported boundary condition and the defining property of a TE mode is equivalent to the set of equations

$$\begin{aligned} \mathrm{E}_y &= 0 \\ \text{and} \qquad \mathrm{H}_x &= 0 \end{aligned} \tag{10.37}$$

which contain the significant statements made by the bounary condition equations 10.32 and 10.33.

We have thus verified equation 10.34 as a necessary and sufficient condition for the satisfaction of the boundary conditions at the particular point on the boundary we were considering. The general boundary is not perpendicular to x everywhere, so we put the result back into the form $\partial \mathrm{H}_z/\partial n = 0$ before generalizing it to other points.

10.8 Solutions for Rectangular Waveguide

Now that we have established the general properties of a TE wave solution, we will explore the details of that solution in the particular geometry of a *rectangular waveguide* illustrated together with a suitable coordinate system in Figure 10.6.

10.8.1 Aspects of the solution

In our exploration we will encounter the following aspects.

- The analysis will be performed in the sinusoidal steady state using complex phasors.
- We will assume a complex propagation constant γ.
- The propagation constant depends on the frequency, the waveguide size, and the mode chosen.
- There is an infinite set of distinct solutions to the problem. Each solution will have its own spatial variation of the electric and magnetic fields and will be called a *mode*.
- At the start we will assume that the waveguide walls to be capable of containing the fields without energy loss.

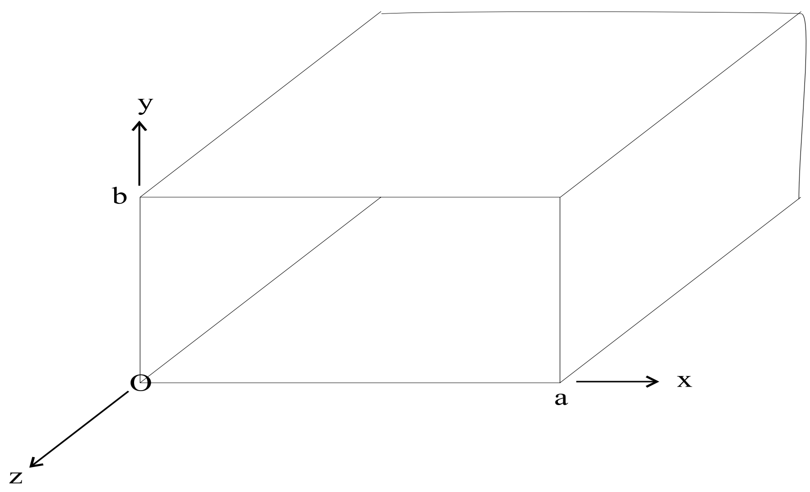


Figure 10.6: Co-ordinate system for rectangular waveguide analysis.

- In this situation the propagation constant γ will be shown to be either purely real or purely imaginary.
- Modes with a purely imaginary propagation constant change phase but not amplitude as we move down the waveguide, and correspond to *propagating waves.*
- Modes with a purely real propagation constant correspond to waves which change in amplitude but not in phase as we proceed down the waveguide. Such waves are called *evanescent waves.*
- Perhaps unexpectedly, the change in amplitude of an evanescent wave does not imply dissipation of energy.
- With each mode is associated a parameter called the *cut off frequency*, the significance of which will be developed.
- One of the modes is called the *dominant mode.* It is the mode with the lowest cut off frequency.
- We will explore the properties of the dominant mode in detail.

10.8.2 Solution procedure

We emphasise again that equations 10.30 and 10.31 are the scalar equations we need to solve. Because the boundaries fit the co-ordinate system rather neatly, it is quite likely we can find a solution by the method of *separation of variables.* The differential equation is

$$\frac{\partial^2 \mathrm{H}_z}{\partial x^2} + \frac{\partial^2 \mathrm{H}_z}{\partial y^2} = -\left[\gamma^2 + \left(\frac{\omega}{c}\right)^2\right] \mathrm{H}_z. \tag{10.38}$$

The z dependence of the fields is $e^{-\gamma z}$ where $\gamma = \alpha + j\beta$. If γ is *imaginary* we have a *propagating mode* and if γ is *real* we have an *attenuating mode*, sometimes called an *evanescent mode*.

For a solution by the method of separation variables, we put

$$\mathrm{H}_z(x, y, z) = F(x)G(y)e^{-\gamma z}. \tag{10.39}$$

Then the equation 10.30 becomes

$$\frac{1}{F}\frac{\partial^2 F}{\partial x^2} + \frac{1}{G}\frac{\partial^2 G}{\partial y^2} = -\left[\gamma^2 + \left(\frac{\omega}{c}\right)^2\right]. \tag{10.40}$$

Since this equation has the form of a function of F alone plus a function of G alone equal to a constant, we must have both functions separately equal to constants. Hence

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2} &= -k_x^2 F(x) \\ \frac{\partial^2 G}{\partial y^2} &= -k_y^2 G(y) \end{aligned} \tag{10.41}$$

where k_x^2 and k_y^2 are constants yet to be found, and from equation 10.40 we have

$$k_x^2 + k_y^2 = \left(\frac{\omega}{c}\right)^2 + \gamma^2. \tag{10.42}$$

The general solution (without having yet applied the boundary coditions) is then

$$\mathrm{H}_z(x, y) = F(x)G(y)e^{-\gamma z} \tag{10.43}$$

where

$$\begin{aligned} F(x) &= F_1 \cos k_x x + F_2 \sin k_x x \\ G(y) &= G_1 \cos k_y y + G_2 \sin k_y y. \end{aligned} \tag{10.44}$$

10.8.3 Exercise

The reader should show as an *exercise* that the boundary conditions on the plane $x = 0$ and $x = a$ require that $F_2 = 0$ and $k_x a$ should be a multiple of say π, say $l\pi$; with l an integer. The boundary conditions on the plane $y = 0$ and $y = b$ require that $G_2 = 0$ and $k_y b$ should be a multiple of π, say $m\pi$, with m an integer.

10.8.4 General solution

Hence the general solution for H_z is

$$\boxed{\mathrm{H}_z = \mathrm{H}\cos\left(\frac{\pi l x}{a}\right)\cos\left(\frac{\pi m y}{b}\right)e^{-\gamma z}}. \tag{10.45}$$

The relation $k_x^2 + k_y^2 = (\omega/c)^2 + \gamma^2$ becomes

$$\boxed{\gamma^2 = -\left(\frac{\omega}{c}\right)^2 + \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}. \tag{10.46}$$

In the above equation a newly introduced amplitude factor H has replaced the product F_1G_1. From these two equations we can derive all we want to know about the TE modes in rectangular waveguide.

10.8.5 General features of the solution

- We have derived quite a lot of solutions, in fact one for each pair of integers l and m.

- The general shape of the field distribution will vary with each combination of l and m. We say each pair defines a *mode* and call the mode the TE_{lm} mode, e.g. TE_{10} mode, TE_{23} mode etc. We will look in a moment at some of the details of the field distributions.

- For each such combination of l and m there is a frequency ω_c defined by

$$\boxed{\left(\frac{\omega_c}{c}\right)^2 = \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \tag{10.47}$$

  below which imaginary values of γ are no longer found. We call ω_c the *cut-off frequency* for that mode. It depends upon the mode numbers l, m, and the dimensions of the waveguide, and is lowest for lower values of l and m.

10.8.6 Dominant mode field

The mode which has the lowest cut-off frequency is called the *dominant mode*. *Below the cut-off frequency for the dominant mode there are no propagating modes, only evanescent modes.*

It is clear from equation 10.47 that when $a > b$ the mode with the lowest cut-off frequency, i.e. the dominant mode, is the TE_{10} mode, which has $l = 1$ and $m = 0$, so that for this mode we have simply

$$\boxed{\mathrm{H}_z = \mathrm{H}\cos\left(\frac{\pi x}{a}\right)e^{-j\beta z}}. \tag{10.48}$$

The remaining field components can be derived from this result using equations 10.4. Note that in those equations our frequently occurring combination of constants $[(\omega/c)^2 +$

$\gamma^2]$ has become $(\omega_c/c)^2$. The longitudinal *magnetic* field is given by equation 10.48, and the longitudinal *electric* field is zero.

10.8.7 Further exercise

The reader should show as an *exercise* that the full set of fields is given by

$$\boxed{\begin{array}{lcl} \mathrm{E}_x &=& 0 \\ \mathrm{E}_y &=& -j\left(\frac{\beta_0}{\beta_c}\right)\eta\ \mathrm{H}\sin\left(\frac{\pi x}{a}\right)e^{-\gamma z} \\ \mathrm{E}_z &=& 0 \\ \mathrm{H}_x &=& j\left(\frac{\beta}{\beta_c}\right)\ \mathrm{H}\sin\left(\frac{\pi x}{a}\right)e^{-\gamma z} \\ \mathrm{H}_y &=& 0 \\ \mathrm{H}_z &=& \mathrm{H}\cos\left(\frac{\pi x}{a}\right)e^{-\gamma z} \end{array}} \tag{10.49}$$

where we have brought down the last equation from above, and we have introduced the free space propagation constant

$$\beta_0 = \omega/c \tag{10.50}$$

.

For a propagating mode in which we are normally interested

$$e^{-\gamma z} = e^{-j\beta z}. \tag{10.51}$$

10.8.8 Field configuration

Examination of the above equations leads to the field configuration shown in Figure 10.7. It is higly recommended that students satisfy themselves that the field epressions given in phasor form in equations 10.49 do lead to this picure of the real time-varying fields at the time $t = 0$.

10.8.9 The wall currents

Examination of the field configuration shown in Figure 10.7 leads to the wall current distribution shown in Figure 10.8. It is higly recommended that students satisfy themselves of the truth of this statement.

10.9 Cut-off Phenomena

10.9.1 Results for rectangular waveguide

We rewrite equations 10.46 and 10.47 in the form

$$\left(\frac{\omega}{c}\right)^2 + \gamma^2 = \left(\frac{\omega_c}{c}\right)^2 \tag{10.52}$$

$$\text{where } \left(\frac{\omega_c}{c}\right)^2 = \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2. \tag{10.53}$$

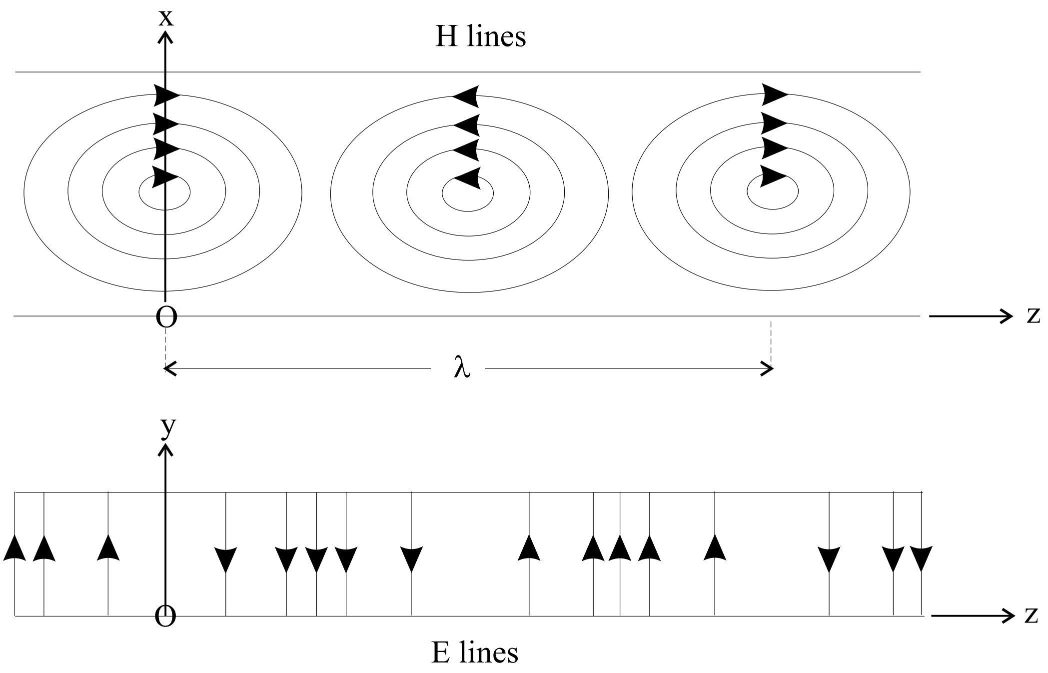


Figure 10.7: Illustration of rectangular waveguide fields.

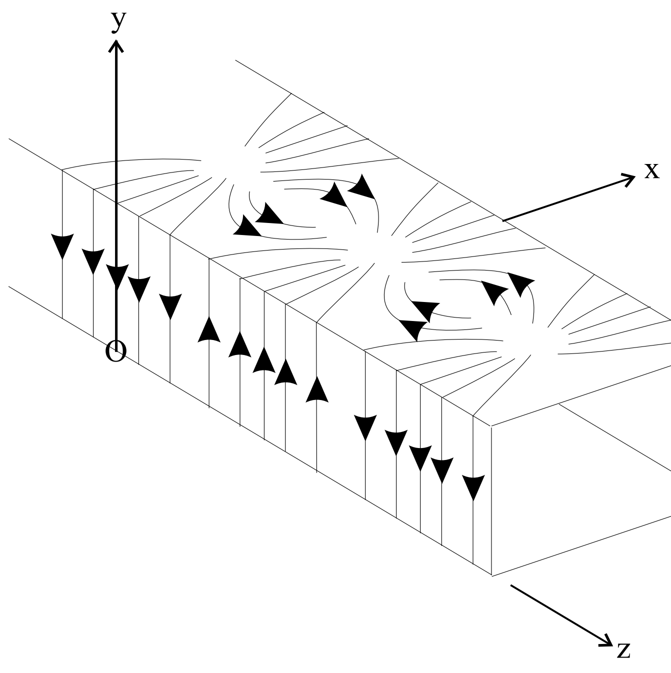


Figure 10.8: Illustration of rectangular waveguide wall currents.

We note first that ω_c varies with the mode. For the present we consider some particular mode.

For imaginary values of γ (real values β) i.e. for *propagating modes*, we must have $\omega > \omega_c$. When $\omega < \omega_c$ we find real values of γ (real values of α) and have *evanescent modes*. Note that we have one *or* the other, and that losses are not considered yet.

10.9.2 Phase and group velocities

The phase and group velocities for a given mode are

$$\boxed{v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}} \tag{10.54}$$

$$\boxed{v_g = \frac{\partial\omega}{\partial\beta} = c\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}} \tag{10.55}$$

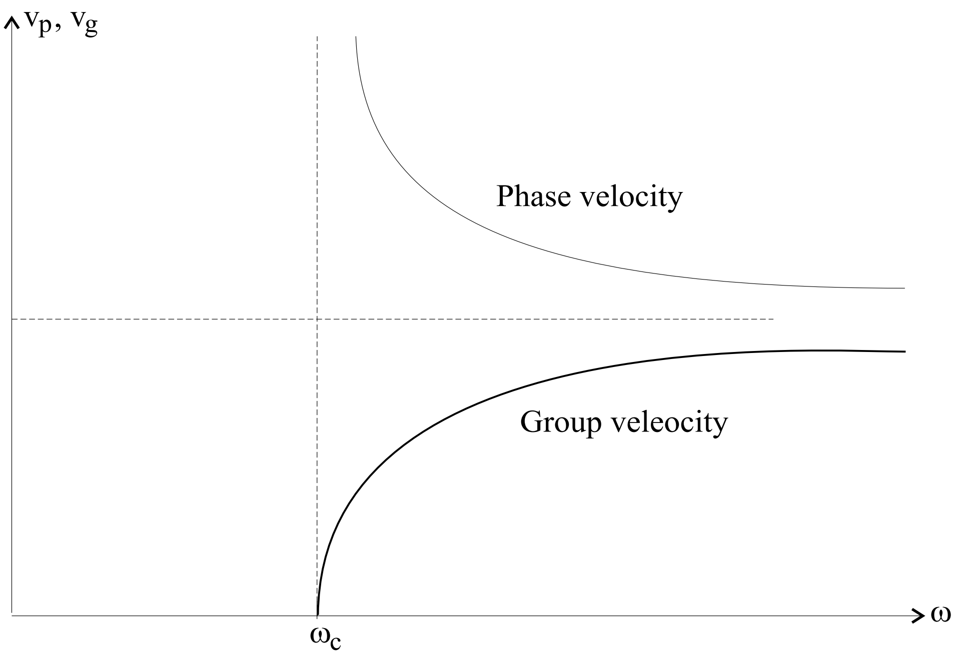


Figure 10.9: Phase and group velocities above cut off.

Students who do not understand the concept of phase and group velocities should request an explanation in lectures. It may already have been given.

An illustration of the behaviour of phase and group velocities is provided in Figure 10.9.

10.9.3 Generalisation

We have established the existence of a cut-off frequency only for TE_{lm} modes in a rectangular waveguide. But in fact for either a *TE mode or a TM mode in any kind of hollow waveguide*, we can show that the propagation vector satisfies an equation of the form

$$\boxed{\left(\frac{\omega}{c}\right)^2 + \gamma^2 = \text{positive constant}} \qquad (10.56)$$

and we then define the constant to be $(\omega_c/c)^2$, as ω_c so defined has the property of a cut-off frequency. Proofs of all these results appear in Reference 1. The important result is that a cut-off frequency always exists and the propagation vector, group and phase velocity are related to it by the above equations and diagrams. Equation 10.53 of course applies only to TE_{lm} modes in rectangular waveguide.

10.9.4 Mode charts

We may list the cut-off frequencies for all the various modes. The relative positions will depend on the aspect ratio of the guide. A display of the result against a frequency axis is called a *mode chart*. Mode charts for two rectancular waveguide *aspect ratios* are shown in Figure 10.10.

10.9.5 Desirable mode charts

In practical situations, for reasons to be discussed in lectures, we desire only one mode to propagate in the frequency band of interest.

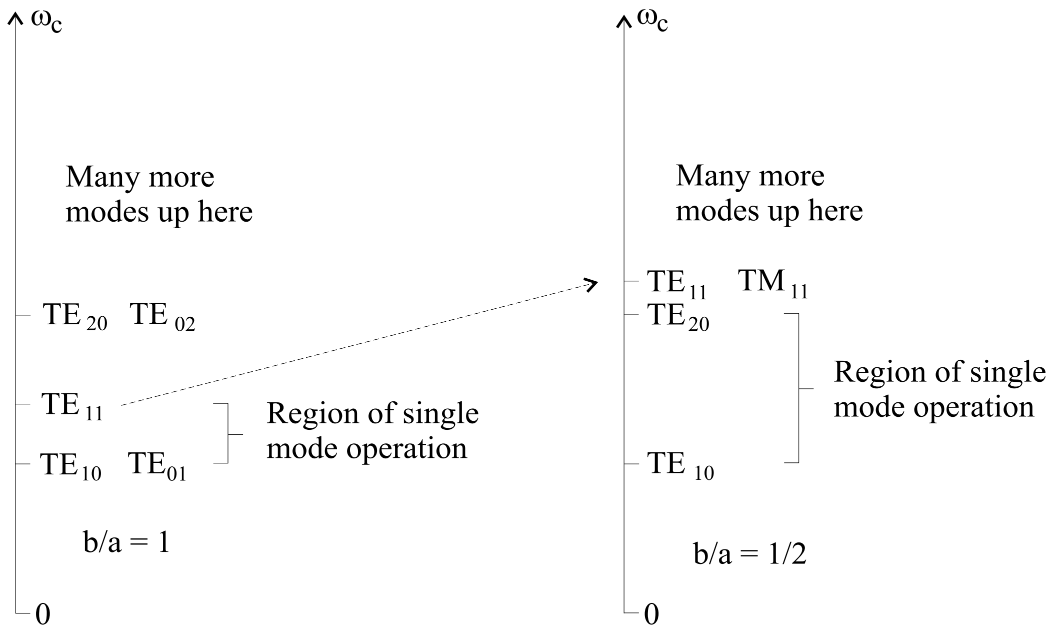


Figure 10.10: Mode chart for rectangular waveguides of different proportions.

The mode chart can be used for finding the useful frequency range of the guide. Rectangular guide as normally constructed has b/a nearly equal to $1/2$, and has a mode chart illustrated in the right of the figure. It may be seen that the dominant mode has its cut-off frequency at half the cut-off frequency of the next highest mode. Thus there is a significant frequency range over which *only one mode will propagate.*

10.9.6 Standard waveguides

The range of standard waveguide sizes shown in Table 10.3 has been defined by the IEC.

Band Designation		Frequency Range	Waveguide Dimensions		Cut off Frequency
			Width	Height	
Common	IEC	GHz	mm	mm	Ghz
L	R-14	1.12-1.70	165.10	82.60	0.91
L_s	R-22	1.70-2.60	109.20	54.60	1.38
S	R-32	2.60-3.95	72.10	34.00	2.08
G	R-48	3.95-5.85	47.60	22.20	3.16
J	R-70	5.85-8.20	34.80	15.80	4.29
X	R-100	8.20-12.40	22.86	10.16	6.50
K_a	R-140	12.40-18.00	15.80	7.90	9.49
K	R-220	18.00-26.50	10.67	4.32	14.10
K_u	R-320	26.50-40.00	7.11	3.56	21.10

Table 10.3: Standard waveguide data chart.

10.9.7 Higher order modes

Sketches of the field configurations and expressions for the fields of quite a number of TE and TM modes can be found in Reference 1.

10.9.8 TE modes with other boundaries

The reader may refer to Reference 1 for details of *circular waveguide modes*. It is useful to look at the pictures of the field distributions, but detailed study is not required. It is particulary useful to observe how each mode strictly obeys the *boundary conditions* of zero tangential electric field and zero normal magnetic flux density at the waveguide walls.

10.10 Transverse Magnetic Modes

Detailed analysis of the properties of transverse Magnetic (TM) modes may be found in Reference 1. The results largely parallel those of the case of TE modes already discussed, with one important difference pointed out at the end of this section. A summary only will be presented here.

The defining property of the modes is

$$\begin{aligned} \mathrm{H}_z &= 0 \\ \mathrm{E}_z &\neq 0. \end{aligned} \tag{10.57}$$

We can derive all fields from the single longitudinal field E_z via equations 10.4. The equation satisfied by E_z is

$$\nabla^2_{xy}\ \mathrm{E}_z = -\left[\left(\frac{\omega}{c}\right)^2 + \gamma^2\right]\ \mathrm{E}_z. \tag{10.58}$$

We can show all the boundary conditions are contained in the single equation

$$\mathrm{E}_z = 0 \qquad \text{at the boundary.} \tag{10.59}$$

We find a comparable sequence of modes to the above TE modes. All of the fields can be derived from the generator field

$$\boxed{\mathrm{E}_z = \ \mathrm{E}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)}. \tag{10.60}$$

We note that if l or $m = 0$, then *all fields vanish.* This property of TM modes is in marked contrast to the behaviour of TE modes. The propagation constant is related to mode number by the equation

$$\boxed{\gamma^2 = -\left(\frac{\omega}{c}\right)^2 + \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \tag{10.61}$$

which is the *same* as equation 10.46 for TE modes. Note that the TM modes are thus *degenerate* with the TE modes for all l, m, except that if $l = 0$ or $m = 0$, the TM mode does not exist. It is this last property that allows the dominant mode in rectangular waveguide to be a TE mode unaccompanied by any degenerate TM mode.

Chapter 11

INTRODUCTION TO RADIATION

We now turn to a study of the mechanism whereby electromagnetic waves may be used to convey power or information from one location to another without the participation of intervening wires or other wave guiding structures.

This will require the study firstly of how electromagnetic waves may be launched by currents and charges in a system of conductors called a *transmitting antenna* which is connected to a *transmitter*, and how at another location at a significant distance those electromagnetic waves may induce voltages and currents in a system of conductors and forming what is called a *receiving antenna* and which is connected to an electronic circuit known as a *receiver*. Such a transmission system is shown schematic form in Figure 11.1.

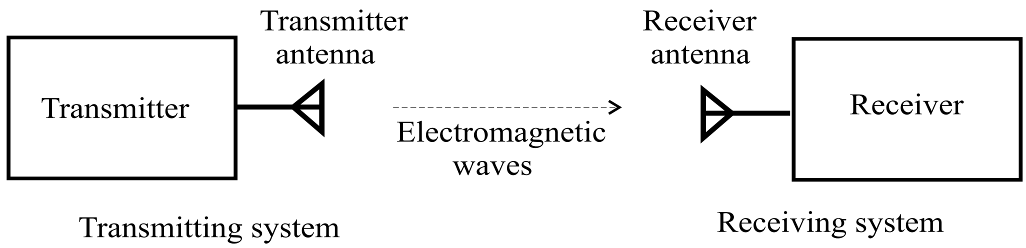


Figure 11.1: Communication of information or power by electromagnetic waves.

In the study below most of the attention will be on the *transmitting situation*, but the *receiving situation* will also receive appropriate attention.

The treatment will be introductory and use will be made of some advanced theoretical results which will be introduced without proof. Among these results is that there is a strong relation between the transmitting and receiving situations which derives from a fundamental theorem of electromagnetic theory known as the *Lorentz reciprocity theorem*. Detailed study of that theorem is left for a following year, but the conclusions which may be drawn from it in the radiation context will be stated and used in this chapter.

11.1 Introduction

11.1.1 The transmitting problem

The essence of the transmitting problem is illustrated in Figure 11.2 in which a known distribution of charges and currents in the vicinity of the origin, or a known distribution of electric and magnetic fields in an aperture near the origin, produces at a great distance electromagnetic fields which to a very good approximation take the form of the simple plane waves discussed earlier in Chapter 8. The problem is the calculation of those fields at great distance from the known distribution of charge and current sources, or of the fields in the aperture.

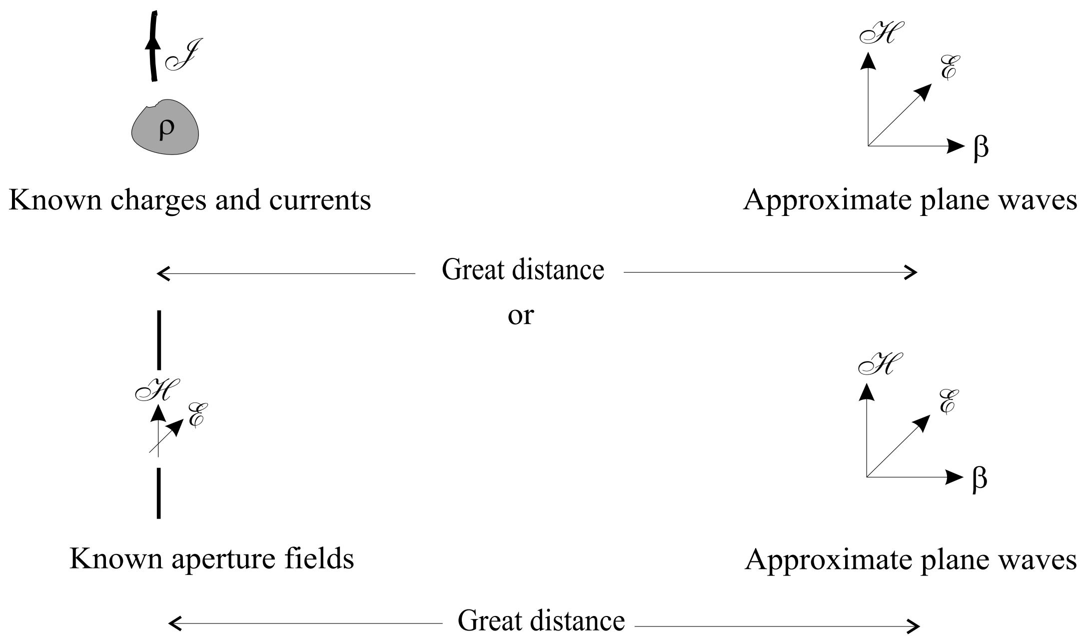


Figure 11.2: The transmitting situation.

11.1.2 Examples of antennas

Four simple examples of electromagnetic antennas are shown in Figure 11.3. As we will see later, each of these antennas may be used in either a transmitting or receiving role. The transmission line fed dipole or transmission line fed current loop are examples of antennas where the known quantities are charges or currents on the conductors. The slots in waveguide and waveguide fed horn are examples of antennas where the known quantities are fields in an aperture.

11.1.3 Radiation questions

Among the questions to which we would like answers in the radiation problem are the following

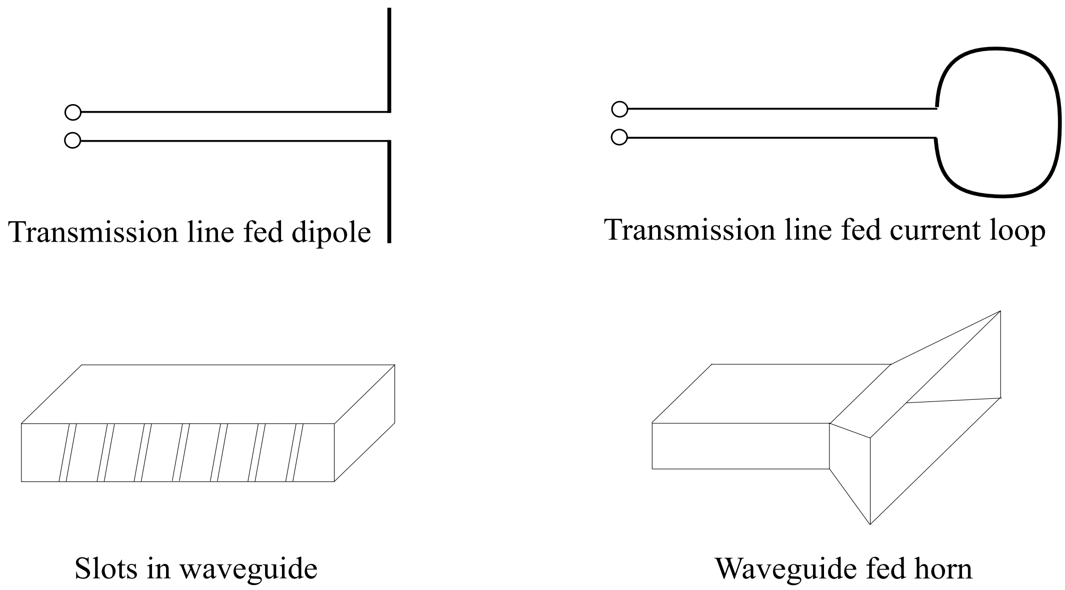


Figure 11.3: Simple examples of antennas.

- What are the fields which the antenna can create at considerable distance?
- How do these fields vary with direction from the antenna?
- What is the total power radiated by the antenna?
- What is the input impedance of the antenna?
- What is the frequency variation of this input impedance?
- What is the efficiency of the antenna?
- How does the antenna behave in its receiving mode?

11.1.4 Scope of treatment

In this introductory treatment, we will be able to obtain answers to only some of these questions. Our main objective is to introduce the very powerful concept of *retarded potentials*, and to see how they might be applied to the radiation problem.

It is important to note that we will attempt to calculate by this method only the fields at considerable distance, and will be prepared to make suitable approximations in doing so. It might be noted, however, that the electromagnetic fields at points close to the antenna can also be calculated from the retarded potentials, but in that calculation the above mentioned approximations may not be made, and the mathematical labour is thus considerably increased.

11.1.5 Procedure for calculation

- We will work in terms of the *sinusoidal steady state*.

- We work via the method of *retarded potentials* to be introduced below.
- Our calculation method will focus mainly upon the currents and not the charges. This is because in the sinusoidal steady state we may easily express the charges in terms of the currents via the *conservation equation*.
- We will consider only the behaviour of antennas containing known charges and currents and leave to later years a study of antennas employing fields in an aperture.
- We assume that the currents which produce the radiation are known.
- We develop, at a suitable time, a system of *approximations* which give for the fields, expressions which are asymptotically correct at large distances, and provide a substantial reduction in algebraic labour over that required for a complete solution.

11.2 Retarded Potentials

11.2.1 Definition

In the time domain the electric scalar potential $\phi(\mathbf{r}_2, t)$ and the magnetic vector potential $\mathsf{A}(\mathbf{r}_2, t)$ produced at time t at a point $\mathbf{r}_2$ by charge and current distributions $\rho(\mathbf{r}_1)$ and $\mathsf{J}\ (\mathbf{r}_1)$ at various points $\mathbf{r}_1$ are given by

$$\boxed{\phi(\mathbf{r}_2, t) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho(\mathbf{r}_1, t - r_{12}/c)}{r_{12}} dv} \tag{11.1}$$

and

$$\boxed{\mathsf{A}(\mathbf{r}_2, t) = \frac{\mu_0}{4\pi} \int_v \frac{\mathsf{J}\ (\mathbf{r}_1, t - r_{12}/c)}{r_{12}} dv}. \tag{11.2}$$

11.2.2 Interpretation

In the above formulae the denominator term r_{12} is the scalar distance between the point $\mathbf{r}_1$ at which the charge or current element exists and the point $\mathbf{r}_2$ at which the potential is to be calculated. It is to be noted that in order to calculate the potential at time t, one has to take as the argument of the integral the value of charge or current density which existed at an earlier time $t - r_{12}/c$.

Since r_{12}/c is the time taken for an electromagnetic wave to propagate from the point $\mathbf{r}_1$ to the point $\mathbf{r}_2$, these equations may be interpreted as saying that each charge element ρdv has associated with it a scalar potential ϕ which propagates out in all directions at the speed of light, and diminishes as the first power of the scalar distance between the source and the point at which the potential is being calculated, and each current element $\mathsf{J}\ dv$ has an associated magnetic vector potential element $d\mathsf{A}$, parallel to the current element, and which also propagates out in all directions at the speed of light and diminishes as the first power of the scalar distance between the source and the point at which the potential is being calculated.

It is to be noted that, if there be no time variation or if the retardation effect be neglected, the above formulae reduce to the formulae for electrostatic scalar potential and magnetostatic vector potential hopefully studied in Level Two electromagnetic theory courses. As it is possible that this material has been forgotten, the lecturer should be requested to provide a short revision.

11.2.3 Calculation of field vectors

The formulae by means of which the electric field intensity and magnetic flux density may be calculated from these potentials are

$$\boxed{\mathsf{B} = \textbf{curl}\ \mathsf{A}} \tag{11.3}$$

$$\boxed{\mathsf{E} = -\textbf{grad}\ \phi - \frac{\partial \mathsf{A}}{\partial t}}. \tag{11.4}$$

A plausibility argument for the formulae will be given in lectures. Proofs can be found in text books and in notes for Level 4 courses of this Department.

11.2.4 Sinusoidal steady state forms

Most radiation problems are solved in the sinusoidal steady state. In that case the above formulae become considerably simplified through the fact that evaluation of the charge and current density *at an earlier time* $t - r_{12}/c$ is equivalent to the multiplication by a *phase retardation factor* $e^{-j\beta r_{12}}$ where β is the propagation constant for electromagnetic waves at the angular frequency ω of the excitation. Thus the above equations for the potentials become

$$\boxed{\phi(\mathbf{r}_2) = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho(\mathbf{r}_1)e^{-j\beta r_{12}}}{r_{12}}dv} \tag{11.5}$$

$$\boxed{\mathbf{A}(\mathbf{r}_2) = \frac{\mu_0}{4\pi}\int_v \frac{\mathbf{J}(\mathbf{r}_1)e^{-j\beta r_{12}}}{r_{12}}dv} \tag{11.6}$$

where all quantities now appearing are *complex phasors* and are no longer time dependent. The expressions for the electric field intensity and magnetic flux density become in phasor terms

$$\boxed{\mathbf{B} = \textbf{curl}\ \mathbf{A}} \tag{11.7}$$

and

$$\boxed{\mathbf{E} = -\textbf{grad}\ \phi - j\omega\mathbf{A}}. \tag{11.8}$$

11.2.5 Connections between ϕ and A

The development so far has been made it appear that the electric scalar potential and magnetic vector potential may be independent of one another. There is however a connection between the charge and current density expressed by the *charge conservation equation* discussed in earlier chapters and reproduced for convenience below, firstly in the situation of arbitrary time dependence as

$$\text{div } \mathsf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{11.9}$$

and in the case of the sinusoidal steady state as

$$\text{div } \mathbf{J} + j\omega\rho = 0. \tag{11.10}$$

Because ρ and J are related by the charge conservation equation, ϕ and A as we have defined them are also related, in fact in the time domain by the equation

$$\boxed{\text{div } \mathsf{A} + \mu_0\epsilon_0\frac{\partial \phi}{\partial t} = 0} \tag{11.11}$$

and in the sinusoidal steady state by the equation

$$\boxed{\text{div } \mathbf{A} + j\omega\mu_0\epsilon_0\phi = 0.}. \tag{11.12}$$

We will not prove these relations but the similarity with the conservation equation may be noted. On account of these relations it is possible to obtain both the electric field intensity and the magnetic flux density from a knowledge of the magnetic vector potential alone. This calculation procedure can be followed for both the near and far fields, but we will in the next section adopt an even simpler method which is suitable for the calculation of only far-field terms.

To see how the electric scalar potential may be eliminated from the calculation we note that we have from equation 11.12 and assuming that $\omega \neq 0$,

$$\phi = -\frac{\text{div } \mathbf{A}}{j\omega\mu_0\epsilon_0} \tag{11.13}$$

substituting for ϕ in equation 11.8 and bringing forward equation 11.7 in a form appropriate to free space operation, we have

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{curl}\ \mathbf{A} \tag{11.14}$$

$$\begin{aligned} \mathbf{E} &= \frac{1}{j\omega\mu_0\epsilon_0}\mathbf{grad}\ \text{div } \mathbf{A} - j\omega\mathbf{A} \\ &= -\frac{j\omega}{\beta^2}\mathbf{grad}\ \text{div } \mathbf{A} - j\omega\mathbf{A} \end{aligned} \tag{11.15}$$

where we have used the commonly occurring relations $c = 1/\sqrt{\mu_0\epsilon_0}$ and $\omega = c\beta$.

We can see from this pair of equations that both the electric field and the magnetic field can be derived from the magnetic vector potential, which in turn can be derived from the current distribution on the antenna. It is for this reason that radiation calculations for many antennas *proceed from the antenna current alone.*

11.2.6 Near and far fields

If we require the electromagnetic fields at all positions in relation to the antenna, we are generally faced with a significant amount of algebra, involving either the calculation of both potentials, or the calculation of the magnetic potential and the performance of two orders of spatial differentiation thereon.

If, however, we are content to ask questions only about the field components at considerable distance, at which point we may be reasonably assured that the relationship between the electric and magnetic field intensities is simply that for uniform plane waves, we may perform the calculation of the magnetic flux density **B** from equation 11.7, and employ simple plane wave concepts to derive the associated electric field intensity **E**.

11.3 The Short Dipole

We now embark upon the study of the transmission characteristics of a particular antenna in the form of a straight wire, carrying an oscillatory current, and whose length is much less than the electromagnetic wavelength at the operating frequency. Such an antenna is called a *short electric dipole*.

Although it might at first appear to be incongruous, we will assume that the current has the same value over the full length of the antenna. An argument will be provided in lectures to justify the making of this apparently physically unrealisable assumption.

11.3.1 Specification of the problem

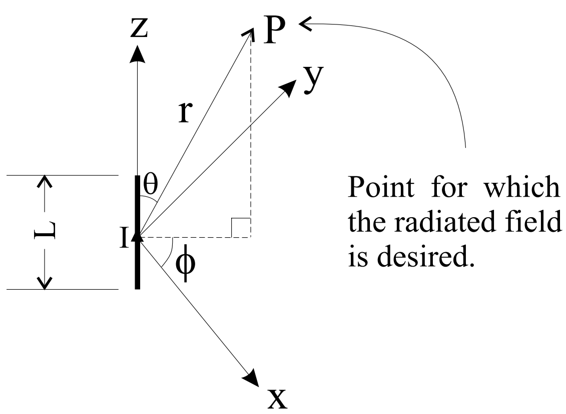


Figure 11.4: Co-ordinate system for short dipole analysis.

We will calculate here the field radiated by a short dipole consisting of an oscillating current having magnitude I flowing uniformly along a conductor of vector length **L**, as shown in Figure 11.4.

As it will later be found that the strengths of the radiated fields are proportional to the product of the current and the length, we will tke the opprtunity to define a quantity which we will call the strength **P** of the dipole by the equation

$$j\omega\mathbf{P} = \mathrm{I}\mathbf{L} \tag{11.16}$$

Under this definition the dipole strength $\mathbf{P}$ just defined has the same units as has an *electrostatic dipole*, namely Cm. It should be noted that in some treatments of radiation a dipole strength with different units is sometimes defined, and care in the interpretation of such other treatments is therefore required.

11.3.2 Change of co-ordinates

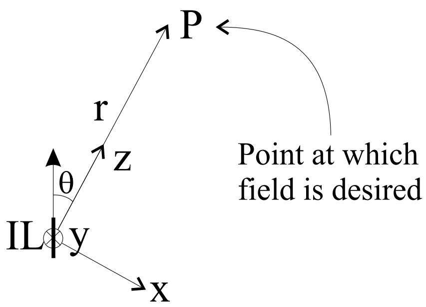


Figure 11.5: Rotated co-ordinates at point P.

In Figure 11.4 is shown the system of co-ordinates in which it is most natural to present the problem, and in which we would eventually wish to obtain the results. However to continue the derivation in these co-ordinates we would require expressions for div, **grad** etc. in spherical polar co-ordinates.

We will avoid the complexity of this approach by choosing, without loss of generality, the co-ordinate system illustrated in Figure 11.5, in which the dipole is located at the origin, the z axis is chosen along the direction OP, and the x axis is chosen to lie in the plane defined by the z axis and the vector $\mathbf{P}$. Note that this is a new xyz system from the one used in Figure 11.4.

The components of the dipole vector in these co-ordinates are

$$\mathbf{P} = \begin{bmatrix} \mathrm{P}_x \\ 0 \\ \mathrm{P}_z \end{bmatrix} = \begin{bmatrix} -\mathrm{P}\sin\theta \\ 0 \\ \mathrm{P}\cos\theta \end{bmatrix} \tag{11.17}$$

.

11.3.3 Analysis of potential

The retarded vector potential is then

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_v \frac{\mathbf{J}e^{-j\beta z}}{z}dv \tag{11.18}$$

where we have used $\beta = \omega/c$. We now replace $\int \mathbf{J}dv$ by $\mathrm{I}\mathbf{L} = j\omega\mathbf{P}$ and obtain

$$\mathbf{A} = \frac{\mu_0}{4\pi}(j\omega\mathbf{P})\frac{e^{-j\beta z}}{z} \tag{11.19}$$

The first factor is a constant independent of position. The second and third factors provide the electromagnetic field components when we perform various vector differential operations on them. Before doing this however we note the following properties of the expected algebra.

- Two spatial variation factors are present.
- Two orders of spatial differentiation are to be preformed.
- We anticipate a large amount of algebra to arise from the two aspects mentioned above.
- The fields set up by an oscillating dipole are therefore quite complicated. In fact we expect $1/z$, $1/z^2$, and $1/z^3$ terms to occur as a result of the operations indicated above.
- All the fields diminish with distance. We only want the $1/z$ (i.e. $1/r$) terms in a study of the far field, as only these have significant amplitude at large distances.
- These terms will be relatively easy to calculate for a number of reasons set out below.

  Firstly we may neglect in their derivation all operations wherein we differentiate the denominator, because such differentiation will only produce terms with higher powers of z in the denominator and which become progressively more negligible as the distance $r = z$ increases.

  Secondly any differentiation with respect to the x or y coordinates produces at large distance negligible variations in the polar coordinates θ and ϕ, and terms deriving from differentiation with respect to the x or y co-ordintes as well become negligible.

  Finally, differentiation with respect to z of the numerator factor $e^{-j\beta z}$ is easy; we just multiply by $-j\beta$.
- Later in Section 11.4, we will generalise these ideas.
- Another way of expressing these ideas is that at large distance the variation of the potentials as a result of the variation of the *phase delay* between the position of the source and the position at which the fields are desired so much dominates all other forms of variation of the vector potential that they may be neglected. We are beginning to see the absolutely crucial role that *phase delay effects* play in the radiation problem.
- These matters will hopefully receive thorough discussions in lectures.

With these observations to guide us we proceed to evaluate

$$
\begin{aligned}
\mathbf{curl\ A} &\approx \frac{j\omega\mu_0}{4\pi z}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{P}_x e^{-j\beta z} & 0 & \mathrm{P}_z e^{-j\beta z} \end{vmatrix} \\
&= \frac{-j\omega\mu_0}{4\pi z}\begin{bmatrix} 0 \\ j\beta \mathrm{P}_x e^{-j\beta z} \\ 0 \end{bmatrix}. \qquad (11.20)
\end{aligned}
$$

.

Thus the radiation component of the magnetic field has a y component only given by

$$\mathrm{H}_y = -j\beta j\omega \frac{\mathrm{P}_x e^{-j\beta z}}{4\pi z}. \tag{11.21}$$

We note that this fits our expectation of an approximately uniform plane wave. Next we calculate the electric field, evaluating first

$$\mathrm{div}\mathbf{A} \approx \frac{\partial \mathrm{A}_z}{\partial z} = \frac{j\omega\mu_0 \mathrm{P}_z(-j\beta)e^{-j\beta z}}{4\pi z} \tag{11.22}$$

and then

$$\mathbf{grad}\,\mathrm{div}\,\mathbf{A} = \frac{\mu_0 j\omega \mathrm{P}_z(-j\beta)}{4\pi z}\begin{bmatrix} 0 \\ 0 \\ (-j\beta)e^{-j\beta z} \end{bmatrix}. \tag{11.23}$$

The first term we require for the electrical field is thus

$$\frac{-j\omega}{\beta^2}\mathbf{grad}\,\mathrm{div}\,\mathbf{A} = \frac{-\omega^2\mu_0 e^{-j\beta z}}{4\pi z}\begin{bmatrix} 0 \\ 0 \\ \mathrm{P}_z \end{bmatrix}. \tag{11.24}$$

The second term we require for the electric field is simply

$$-j\omega\mathbf{A} = \frac{-\omega^2\mu_0 e^{-j\beta z}}{4\pi z}\begin{bmatrix} -\mathrm{P}_x \\ 0 \\ -\mathrm{P}_z \end{bmatrix}. \tag{11.25}$$

The electric field is the sum of these two terms. It may be seen that the z components cancel, and we are left with only an x component of field given by

$$\mathrm{E}_x = \frac{\omega^2\mu_0 \mathrm{M}_x e^{-j\beta z}}{4\pi z}. \tag{11.26}$$

We note that this expression also fits our expectation of an approximately uniform plane wave. The ratio of electric to magnetic field amplitudes is

$$\frac{\mathrm{E}_x}{\mathrm{H}_y} = \frac{\mu_0\omega^2}{\beta\omega} = \mu_0\frac{\omega}{\beta} = \mu_0 c = \mu_0\sqrt{\frac{1}{\mu_0\epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \eta \tag{11.27}$$

as expected for a uniform plane wave.

11.3.4 Return to polar co-ordinates

We will now translate the field components we derived into the spherical polar co-ordinates in which the problem was first posed and which better reflect the symmetry of the results. Figure 11.6 shows the relation between the original co-ordinate system and the one most recently used.

Since $\mathrm{P}_x = -\mathrm{P}\sin\theta$ we have

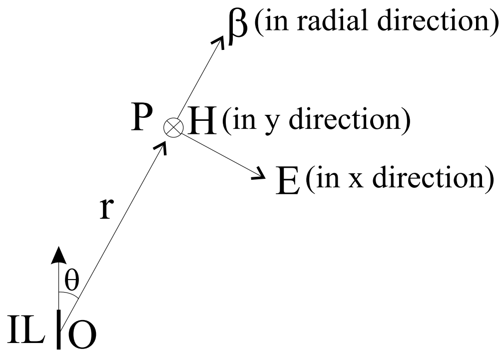


Figure 11.6: Far field components radiated by short electric dipole.

$$\begin{aligned} \mathrm{E}_\theta &= \mathrm{E}_x = \frac{\omega^2\mu_0 \mathrm{P}\sin\theta e^{-j\beta r}}{4\pi r} \\ \text{and} \qquad \mathrm{H}_\phi &= \mathrm{H}_y = \frac{-\omega\beta \mathrm{P}\sin\theta e^{-j\beta r}}{4\pi r}. \end{aligned} \tag{11.28}$$

11.3.5 Radiated power

The Poynting vector $\frac{1}{2}(\mathbf{E}\times\mathbf{H}^*)$ is in the $\mathbf{r}$ direction and has the value

$$\mathrm{S}_r = \mathrm{S}_z = \frac{\mu_0\omega^3\beta|\mathbf{P}|^2\sin^2\theta}{2(4\pi r)^2}. \tag{11.29}$$

This vector, being real, gives the real power per unit area flowing across an element of area perpendicular to $\mathbf{r}$ at a great distance.

11.3.6 Radiation pattern

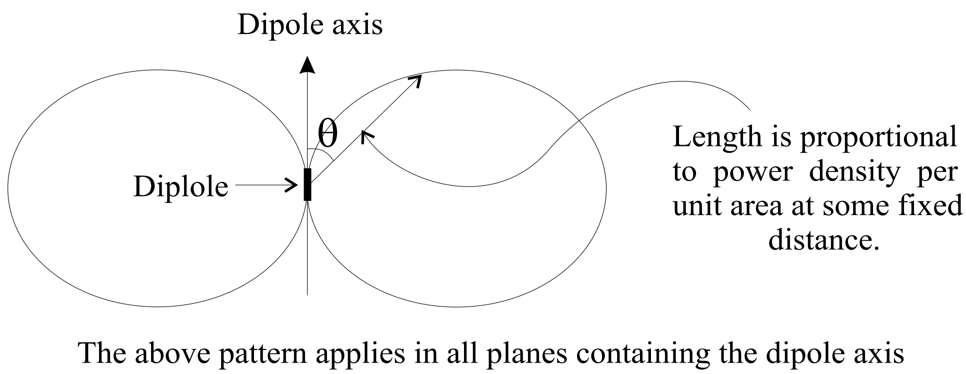


Figure 11.7: Radiation pattern of a short electric dipole.

From the expression 11.29 for the radiated power density in various directions it is possible to construct the Polar Diagram of Figure 11.7 which shows the intensity of radiation in various directions.

We observe that *no radiation takes place along the dipole axis*, and the radiation pattern has *axial symmetry*, with *maximum radiation being in the equatorial plane.* We will show later that we see the same sort of pattern for the receiving behaviour of an antenna.

11.3.7 Antenna gain

Because of the non-uniform nature of the pattern we have the concept of *antenna gain*, which is defined *for a lossless antenna* as the power flow per unit area for this antenna in the most efficient direction over the power flow per unit area we would obtain if the energy were uniformly radiated in all directions. To calculate the gain we need calculate first the total radiated power

$$\begin{aligned} W &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \Re e\left\{\mathrm{S}_r\right\}\left(r^2\sin\theta\, d\theta\, d\phi\right) \\ &= \frac{\mu_0\omega^3\beta|\mathbf{P}|^2}{32\pi^2}\int_{\theta=0}^{\pi}\sin^3\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi \\ &= \frac{\mu_0\omega^3\beta|\mathbf{P}|^2}{12\pi}. \end{aligned} \tag{11.30}$$

.

From this we calculate average radiated power per unit area, i.e.

$$\frac{W}{4\pi r^2} = \frac{\mu_0\omega^3\beta|\mathbf{P}|^2}{48\pi^2 r^2}. \tag{11.31}$$

.

Hence the Antenna Gain, g defined by

$$\boxed{g = \frac{\text{radiated power/unit area in the most efficient direction}}{\text{average radiated power/unit area over a large sphere}}} \tag{11.32}$$

becomes

$$g = \frac{\omega^3\beta|\mathbf{P}|^2}{32\pi^2 r^2}\frac{48\pi^2 r^2}{\omega^3\beta|\mathbf{P}|^2} = \frac{3}{2}. \tag{11.33}$$

This result is the gain of a small dipole. For more complicated antennas the definition of g is as above but the calculation is more involved, and the results are different. The more involved calculations can most conveniently be performed after we have studied in Section 11.4 a systematic series of approximations for calculating the far-fields for more complex radiating systems.

11.3.8 Radiation resistance

We recall the result 11.30 in the form

$$W = \frac{\mu_0\omega^3\beta|\mathbf{P}|^2}{12\pi} = \frac{\mu_0\omega\beta|\mathrm{I}|^2 L^2}{12\pi}. \tag{11.34}$$

This resistance R_r is defined as the equivalent resistance which would absorb the same power W from the same current I, i.e.

$$W = \frac{R_r|\mathrm{I}|^2}{2}. \tag{11.35}$$

Combining these results we obtain

$$\boxed{R_r = \frac{\mu_0 \omega \beta L^2}{6\pi}}. \tag{11.36}$$

Using the familiar results $\omega = c\beta$, $\beta = 2\pi/\lambda$, $c = 1/\sqrt{\mu_0\epsilon_0}$ and $\eta = \sqrt{\mu_0/\epsilon_0}$, we find

$$R_r = \frac{\eta}{6\pi}(\beta L)^2 = \left(\frac{2\pi}{3}\right)\eta\left(\frac{L}{\lambda}\right)^2 \tag{11.37}$$

If we make the usual substitution $\eta \approx 120\pi\Omega$ we obtain

$$\boxed{R_r \approx 20(\beta L)^2\,\Omega}. \tag{11.38}$$

From this result the following observations can be made

- The radiation resistance of a short dipole is a small fraction of the characteristic impedance of free space, η, which has the value of approximately 377Ω.
- We therefore expect short dipole radiation resistances to be just a few ohms.
- These radiation resistances are often difficult to match efficiently to signal sources, particularly as it happens that the small radiation resistance is in series with a large reactance.
- Apart from a possible matching problem, the *efficiency* of a short dipole antenna will decrease with L. This is because series resistance loss $\propto L$ and radiation resistance $\propto L^2$.

These matters should be expanded upon in lectures.

11.4 Systemisation of Radiation Calculations

With the above analysis of a small electric dipole as background, we now turn to consideration of how we will calculate the radiation from an arbitrary distribution of charges and currents on an antenna. We will not in this analysis seek a complete solution for the electromagnetic fields both close to and far from the antenna. Instead we will be concerned with seeing how the fields at great distance from the antenna may be calculated by using approximations which are similar to those employed in the above analysis.

11.4.1 Co-ordinate system

The arbitrary system of radiating currents which we study is illustrated in Figure 11.8. Here the region of antenna currents is restricted to a limited region of space surrounding an arbitrarily placed origin. A representative current element is shown at the point P_1 whose position vector relative to that origin is $\mathbf{r}_1$. The position vector $\mathbf{r}$ of the field point P_2 relative to the origin is $\mathbf{r}_2$. A spherical polar coordinate system is introduced to further define the position of the far field point clearly. The point P_2 at which the fields are desired is well outside of the shaded region, and is therefore far from all antenna currents. The angle ψ is the angle between the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. The vector $\mathbf{r}_{12}$ gives the position of the far field point P_2 relative to the arbitrary antenna current element P_1. Finally, the angle α is that between the vectors $\mathbf{r}_{12}$ and $\mathbf{r}_2$.

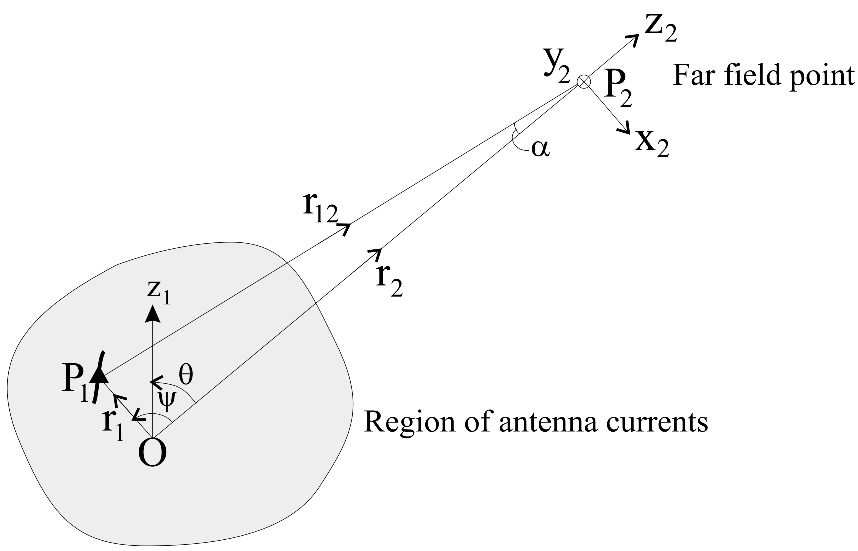


Figure 11.8: Co-ordinates for systemisation of radiation calculations.

11.4.2 Features of the analysis

All fields come from the vector potential

$$\mathbf{A}(\mathbf{r}_2) = \frac{\mu_0}{4\pi} \int_v \frac{\mathbf{J}(\mathbf{r}_1) e^{-j\beta r_{12}}}{r_{12}} dv \tag{11.39}$$

To obtain the field from $\mathbf{A}$ we perform various differential operations. As before we note that in $\mathbf{A}$ there are two spatially varying factors, viz $1/r_{12}$ and $e^{-j\beta r_{12}}$.

11.4.3 Approximations

We develop here a series of approximations which will give exact expressions for the $1/r$ terms in the fields at large distances. The discussion in Reference 1 may usefully be consulted to provide additional perspective.

1. For the same reasons as discussed in the short dipole analysis, we do not need to consider the factor $1/r_{12}$ as a variable in these operations.

2. Since we regard r_{12} as fixed in the factor $1/r_{12}$, and P_2 is a distant point, we can replace r_{12} in this factor by the representative value r_2, and as this value is a constant it may be moved outside the integral sign. These approximations lead to

$$\mathbf{A}(\mathbf{r}_2) = \frac{\mu_0}{4\pi r_2}\int_v \mathbf{J}(\mathbf{r}_1)e^{-j\beta r_{12}}dv \tag{11.40}$$

3. Approximations for r_{12} in the factor $e^{-j\beta r_{12}}$ require more care, since phase differences in radiation effects are crucial. The approximation we use is that

$$\begin{aligned} \mathbf{r}_2 &= \mathbf{r}_1 + \mathbf{r}_{12} && (11.41)\\ \text{i.e.}\quad r_2 &\approx r_1\cos\psi + r_{12} \\ \text{i.e.}\quad r_{12} &\approx r_2 - r_1\cos\psi && (11.42) \end{aligned}$$

   This approximation makes use of the fact that the angle α subtended at P_2 by any element of the radiation system is always small, but places no restriction on the angle ψ.

With these approximations we have

$$\mathbf{A}(\mathbf{r}_2) = \frac{\mu_0 e^{-j\beta r_2}}{4\pi r_2}\int_v \mathbf{J}(\mathbf{r}_1)e^{+j\beta r_1\cos\psi}dv \tag{11.43}$$

The absence of the minus sign in the exponent should be noted. We see that in this expression the factor $e^{+j\beta r_1 cos\psi}$ expresses the phase *advance* of the radiation from the element at P_1 relative to the phase it would have had if that element had been positioned at the origin. The above expression is of the form

$$\mathbf{A}(\mathbf{r}_2) = \frac{\mu_0 e^{-j\beta r_2}}{4\pi r_2}\mathbf{R} \tag{11.44}$$

where

$$\mathbf{R} = \int_v \mathbf{J}(\mathbf{r}_1)e^{j\beta r_1\cos\psi}dv \tag{11.45}$$

The vector $\mathbf{R}$ is called the *radiation vector*. It depends on the *internal geometrical distribution* of the currents and on the *direction of P_2 from the origin O*, but not on the distance. The factor

$$\frac{\mu_0 e^{-j\beta r_2}}{4\pi r_2}$$

depends only on the *distance from the origin O to the field point P_2* but not on the internal distribution of the currents in the antenna.

The radiation vector $\mathbf{R}$ can be regarded as an *effective dipole* equal to the sum of individual dipole elements $\mathbf{J}\,dv$, each weighted by phase factor $e^{j\beta r_1\cos\psi}$, which depends on the *phase advance* $\beta r_1 \cos\psi$ of the element in relation to the origin, and in the direction OP_2. It clearly depends upon the direction OP_2.

To obtain **H** from **A** we need **curl A**, or more precisely the $1/r$ terms when we evaluate **curl A** in polar co-ordinates. The various vector differential operations in polar co-ordinates are provided in the Summary of Formulae sheets, from which we obtain

$$\begin{aligned} \mathrm{H}_\theta &= j\beta\frac{e^{-j\beta r}}{4\pi r}R_\phi \\ \text{and} \qquad \mathrm{H}_\phi &= -j\beta\frac{e^{-j\beta r}}{4\pi r}R_\theta \end{aligned} \tag{11.46}$$

These results agree with those obtained earlier for the short dipole, in which R_ϕ was zero as a result of the choice of the co-ordinate system with the polar axis along the axis of the dipole.

The electric fields can be found from the far field relations

$$\begin{aligned} \mathrm{E}_\theta &= \eta\mathrm{H}_\phi \\ \text{and} \qquad \mathrm{E}_\phi &= -\eta\mathrm{H}_\theta \end{aligned} \tag{11.47}$$

or from the more elaborate procedure of performing **grad** div etc on **A**.

11.5 The Small Circular Loop

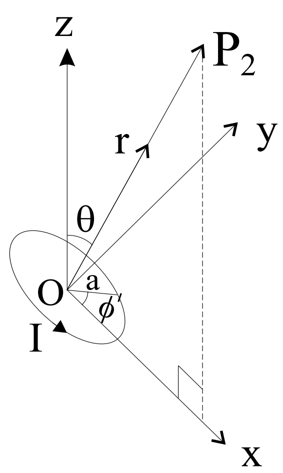


Figure 11.9: Small circular loop in the xy plane.

We now apply the methods of Section 11.4 to the calculation of the fields radiated by a small circular loop of radius a, lying in the xy plane, carrying a phasor current I assumed uniform at all points on the circumference.

11.5.1 Coordinate system

A diagram of the loop, and the co-ordinate system in use, is given in Figure 11.9.

We wish to calculate the radiated fields and power at a large distance.

Because of the symmetry of the problem, the results will be independent of the azimuth co-ordinate ϕ so we may for simplicity set $\phi = 0$.

The point $\mathrm{P}_2(r, \theta, 0)$ is the point at which we wish to calculate the far field. The spherical polar co-ordinates of a point P_1 at a general position on the loop are $(a, \frac{\pi}{2}, \phi')$.

11.5.2 Calculation of the radiation vector

To calculate the contribution to the radiation vector from an element of the loop at the point $\mathrm{P}_1(a, \frac{\pi}{2}, \phi')$, we require $\cos\psi$ where ψ is the angle between OP_1 and OP_2. Since a unit vector in the direction OP_1 has cartesian components $(\cos\phi', \sin\phi', 0)$ and a unit vector in the direction OP_2 has cartesian components $(\sin\theta, 0, \cos\theta)$ we conclude

$$\cos\psi = \sin\theta\cos\phi'. \tag{11.48}$$

The radiation vector is then given by

$$\mathbf{R}(\theta, 0) = \int \mathbf{J}(\mathbf{r}_1) e^{j\beta a \sin\theta\cos\phi^{\theta}} dv. \tag{11.49}$$

It will hopefully be clear that the only component of $\mathbf{R}$ for the direction $(\theta, 0)$ will be the ϕ component, the θ component having vanished by symmetry. This point will be elaborated in lectures. Thus if $\hat{\mathbf{u}}_\phi$ is a unit vector in the ϕ direction we wish to calculate

$$\mathrm{R}_\phi(\theta, 0) = \int \mathbf{J} \cdot \hat{\mathbf{u}}_\phi \, e^{j\beta a \sin\theta\cos\phi^{\theta}} dv. \tag{11.50}$$

Now for a filamentary current the current element $\mathbf{J}dv$ is equivalent to $\mathrm{I}\, d\mathbf{r}_1$, where $d\mathbf{r}_1$ is an element of the current path. Therefore we have

$$\mathrm{R}_\phi(\theta, 0) = \int \mathrm{I}\, d\mathbf{r}_1 \cdot \hat{\mathbf{u}}_\phi e^{j\beta a \sin\theta\cos\phi^{\theta}}. \tag{11.51}$$

At the point $\mathrm{P}_1(a, \frac{\pi}{2}, \phi')$ the component of $d\mathbf{r}_1$ in the ϕ direction is $a\cos\phi'\, d\phi'$. Thus

$$\mathrm{R}_\phi(\theta, 0) = \int_0^{2\pi} \mathrm{I}\, a e^{j\beta a \sin\theta\cos\phi^{\theta}} \cos\phi' d\phi'. \tag{11.52}$$

11.5.3 Approximation for a small loop

To make progress we exploit the fact that the loop radius is small compared with the wavelength, i.e. $\beta a \ll 1$. We may therefore employ a two-term series expansion for the exponential function and obtain

$$\mathrm{R}_\phi(\theta, 0) \approx \mathrm{I}a \int_0^{2\pi} (1 + j\beta a \sin\theta\cos\phi') \cos\phi' d\phi'. \tag{11.53}$$

This integral is easy to perform with the result that

$$\mathrm{R}_\phi(\theta, 0) = j\beta\pi \mathrm{I} a^2 \sin\theta. \tag{11.54}$$

11.5.4 Electric and magnetic field components

The components of electric and magnetic field which may be derived from this are then

$$\mathrm{H}_\theta = \frac{j\beta e^{-j\beta r}}{4\pi r}\mathrm{R}_\phi = \frac{-(\beta a)^2 \mathrm{I}\sin\theta}{4r}e^{-j\beta r} \tag{11.55}$$

$$\mathrm{E}_\phi = -\eta\mathrm{H}_\theta = \frac{(\beta a)^2\eta\mathrm{I}\sin\theta}{4r}e^{-j\beta r}. \tag{11.56}$$

11.5.5 Poynting vector

The power density radiated in this direction is then seen to be

$$\mathrm{S}_r = -\frac{1}{2}\mathrm{E}_\phi\mathrm{H}_\theta^* = \frac{(\beta a)^4\eta|\mathrm{I}|^2\sin^2\theta}{32r^2}. \tag{11.57}$$

11.5.6 Total power radiated

Thus the total power radiated is

$$W = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\mathrm{S}_r r\sin\theta\, d\phi r\, d\theta \tag{11.58}$$

Substitution for S_r from equation 11.57, using the expansion $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$, and performing the interpretation gives

$$W = \frac{\pi\eta|\mathrm{I}|^2(\beta a)^4}{12}. \tag{11.59}$$

11.5.7 Radiation resistance

The radiation resistance may now be obtained from the relation $W = \frac{1}{2}R_r|\mathrm{I}|^2$ and is

$$R_r = \frac{\pi\eta}{6}(\beta a)^4. \tag{11.60}$$

If we make the usual substitution $\eta \approx 120\pi\Omega$ we obtain

$$\boxed{R_r \approx 20\pi^2(\beta a)^4\Omega}. \tag{11.61}$$

11.5.8 Commentary

Several points about radiation from a small current loop can be made. The first is that the fields as given by equations 11.55 and 11.56 have similar functional dependence, viz a $\sin\theta$ variation, to the fields of the small electric dipole as given in equations 11.21 and 11.26. Thus we expect a *similar radiation pattern*, viz no radiation in the polar direction and maximum radiation in the equatorial plane. Secondly we note that the orientations of the electric and magnetic fields have been interchanged; the magnetic field for the loop is in the θ direction whereas for the electric dipole it was in the ϕ direction. Finally we note that the small circular loop is in fact *quite a poor radiator*; its radiation resistance varies as the *fourth power* of βa, a quantity much less than unity, whereas the electric dipole varied as only the *second power* of βL.

11.6 Receiving Behaviour of Antennas

We now have a reasonable understanding of the fields produced by an antenna used as a *transmitter*. We want to know something about the amount of energy which can be recovered from an electromagnetic wave by an antenna used as a *receiver*.

11.6.1 Fundamental results

To gain such knowledge we make use of two fundamental results below.

- The source impedance of an antenna being used as a receiver is the same as its input impedance when used as a transmitter.
- A receiving antenna when matched for maximum power transfer collects from an incident plane wave an amount of energy equal to that crossing an effective area $A_e = \frac{g\lambda^2}{4\pi}$, where g is the gain of antenna when used in its transmitting role.

A detailed discussion of these results may be found in Reference 1.

11.6.2 Theoretical basis

The first result quoted above is a consequence of the linearity of all the equations, and is also necessary to avoid a thermodynamic paradox.

The truth of the second result is less obvious. We can of course always define an effective area A_e by the equation

$$P_r = A_e \times (\text{Power flow/unit area}). \tag{11.62}$$

The substance of the result is in proving the expression for A_e is

$$\boxed{A_{er} = \frac{g_r\lambda^2}{4\pi}}. \tag{11.63}$$

This result is normally established by using the Lorentz Reciprocity Theorem to show that for an arbitrary antenna the effective area is proportional to the antenna gain, and then establishing that the constant of proportionality is $\lambda^2/(4\pi)$ by studying the transmitting and receiveing behaviour of a particular antenna structure, usually a combination of a dipole antenna and a large disk antenna.

11.6.3 Practical application

The practical importance of the results above lies in the calculation of the ratio of received power to transmitted power for a pair of antennas. From our definition of antenna gain we obtain for a point distant r from a transmitter antenna of gain g_t and transmittiig a power P_t the result

$$\boxed{\text{Power flow/unit area} = \frac{g_t P_t}{4\pi r^2}}. \tag{11.64}$$

If we use equation 11.62 to calculate the received power and then use equation 11.63 to substitute for the effective area A_{er} of a receiving antenna in terms of the gain g_r of that antenna when it is used in a transmitting role we obtain

$$\boxed{\frac{P_r}{P_t} = g_r g_t \left(\frac{\lambda}{4\pi r}\right)^2}. \tag{11.65}$$

As an alternative we could substitute for g_t and g_r in terms of A_{et} and A_{er} and obtain

$$\boxed{\frac{P_r}{P_t} = \frac{A_{et} A_{ert}}{\lambda^2 r^2}}. \tag{11.66}$$

The last two equations clearly illustrate the equivalence of the antenna performance in its receiving and transmitting roles.

Appendix A

REFERENCES

A.1 Principal Text

The principal text book for this course is:
Simon Ramo, John R. Whinnery and Theodore Van Duzer, ”Fields and Waves in Communication Electronics”, John Wiley and Sons.

A.2 Useful Reference

Proofs of some of the advanced results can be found in
Richard B. Adler, Lan J. Chu and Robert M. Fano, “Electromagnetic Energy Transmission and Radiation”, John Wiley and Sons.

A.3 Preparatory Texts

Two text books which cover basic material providing a suitable preparation for the present course are

- M. N. O. Sadiku, “Elements of Electromagnetics”, Saunders Publishing.
- W. H. Hayt, “Engineering Electromagnetics”, 5th edition McGraw Hill.

Material on demagnetising factors can be found in:

- J. A. Osborn, “Demagnetising Factors of the General Ellipsoid”, Physical Review, vol 67, pp 351, (1945).

Appendix B

SUMMARY OF BOUNDARY CONDITIONS

The summary of electromagnetic boundary conditions given below is useful as a reminder of the more extensive discussion of that matter provided in Chapter 6 of these notes, and also in the notes for the Fields Section of the Level 2 Fields and Energy Conversion course. Students should be warned, however, that the summary provided here is probably too abbreviated for all the issues canvassed in that larger discussion to be adequately contained, and that familiarity with the detailed analysis provided in the sections referred to is recommended.

SUMMARY OF BOUNDARY CONDITIONS				
Time variation	General case	Intermediate conductivity	Perfect insulator	Very good conductor
d.c.	E_t is continuous	$\leftarrow$	$\leftarrow$	$\mathsf{E}_t = 0$
a.c.	E_t is continuous	$\leftarrow$	$\leftarrow$	$\mathsf{E}_t = 0$
d.c.	$\hat{\mathbf{n}} \times (\mathsf{H}_{t2} - \mathsf{H}_{t1}) = \mathsf{J}_s$	$\mathsf{J}_s = 0$	$\mathsf{J}_s = 0$	
a.c.	$\hat{\mathbf{n}} \times (\mathsf{H}_{t2} - \mathsf{H}_{t1}) = \mathsf{J}_s$	$\mathsf{J}_s = 0$	$\mathsf{J}_s = 0$	$\hat{\mathbf{n}} \times \mathsf{H}_t = \mathsf{J}_s$
d.c.	$\mathsf{D}_{n2} - \mathsf{D}_{n1} = \rho_s$	$\leftarrow$	$\leftarrow$	$\mathsf{D}_n = \rho_s$
a.c.	$\mathsf{D}_{n2} - \mathsf{D}_{n1} = \rho_s$	$\leftarrow$	$\rho_s = 0$	$\mathsf{D}_n = \rho_s$
d.c.	B_n is continuous	$\leftarrow$	$\leftarrow$	$\leftarrow$
a.c.	B_n is continuous	$\leftarrow$	$\leftarrow$	$\mathsf{B}_n = 0$

In the above Table, a left arrow indicates that the statement in the same line under the heading Geneal case is not modified.

Appendix C

SUMMARY OF FORMULAE

The function of this document is to provide a handy collection of formulae useful in the solution of practical problems. It also forms a basis for a formula sheet to be attached to the examination paper, but not all formulae provided here will appear in the examination formula sheet. Those which will not be so supplied are noted herein.

C.1 Physical Constants

The first three physical constants given in this section will be not supplied in the examination sheet.

1. $\mu_0 = 4\pi \times 10^{-7}$ H/m.

2. $\epsilon_0 \approx 8.854$ pF/m.

3. $\eta \approx 120\pi\ \Omega$.

4. The conductivity of copper is approximately 5.8×10^7 S/m.

C.2 Vector Calculus

In spherical polar co-ordinates at point $P(r, \theta, \phi)$ the gradient of a scalar ψ, the divergence of a vector D, and the curl of a vector H are given by

$$\nabla\psi = \frac{\partial\psi}{\partial r}\,\mathbf{a}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\,\mathbf{a}_\phi \tag{C.1}$$

$$\nabla\cdot\mathsf{D} = \frac{1}{r^2}\frac{\partial[r^2 D_r]}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial[D_\theta\sin\theta]}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} \tag{C.2}$$

$$\nabla \times \mathsf{H} = \frac{1}{r\sin\theta}\left[\frac{\partial[H_\phi \sin\theta]}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial[rH_\phi]}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial[rH_\theta]}{\partial r} - \frac{\partial H_r}{\partial\theta}\right]\mathbf{a}_\phi \tag{C.3}$$

C.3 Transmission Lines

None of the results in the nine sub-sections below will be supplied during the examination.

C.3.1 Characteristic Impedance

The characteristic impedance Z_0 of a transmission line of distributed resistance R, inductance L, capacitance C, and conductance G all per unit length is given by

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{C.4}$$

C.3.2 Voltage Reflection factor

The relations between the voltage reflection factor Γ_v at any point on a transmission line and the impedance Z at that point are

$$\Gamma_v(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} \tag{C.5}$$

$$\frac{Z(z)}{Z_0} = \frac{1 + \Gamma_v(z)}{1 - \Gamma_v(z)} \tag{C.6}$$

C.3.3 Transformation Along a Line

The relation between the voltage reflection factor at any point z on a line and that at the origin is, when the forward wave is in the positive direction of the z axis, is:

$$\Gamma_v(z) = \Gamma_v(0)e^{2j\beta z} \tag{C.7}$$

The relation between the voltage reflection factor at the input end of a line and that at the load end of the line is:

$$\Gamma_v(S) = \Gamma_v(L)e^{-2j\beta l} \tag{C.8}$$

C.3.4 Input Impedance of a Line

None of the results given in the following five sub-sections will be provided in the examination.

C.3.5 The General Case

The input impedance Z_I of a general transmission line of characteristic impedance Z_0 and length l terminated in a load impedance Z_L is given by

$$\frac{Z_I}{Z_0} = \frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l} \tag{C.9}$$

C.3.6 Lossless Line Case

The input impedance Z_I of a lossless transmission line of characteristic impedance Z_0 and length l, terminated in a load impedance Z_L, is given by

$$\frac{Z_I}{Z_0} = \frac{Z_L \cos \beta l + j Z_0 \sin \beta l}{Z_0 \cos \beta l + j Z_L \sin \beta l} \tag{C.10}$$

C.3.7 Short Circuit Lossless Line

The input impedance Z_I and input admittance Y_I of a lossless transmission line of characteristic impedance Z_0 and length l, open circuited at its load end, are given by

$$Z_I = jZ_0 \tan \beta l \qquad \text{and} \qquad Y_I = -jY_0 \cot \beta l \tag{C.11}$$

C.3.8 Open Circuit Lossless Line

The input impedance Z_I and input admittance Y_I of a lossless transmission line of characteristic impedance Z_0 and length l, open circuited at its load end, are given by

$$Z_I = -jZ_0 \cot \beta l \qquad \text{and} \qquad Y_I = jY_0 \tan \beta l \tag{C.12}$$

C.3.9 Quarter Wave Transformers

For a quarter wave transformer, the relation between the input impedance ZI and the load impedance Z_L is

$$Z_I = \frac{Z_0^2}{Z_L} \tag{C.13}$$

C.4 Co-axial Lines

C.4.1 Co-axial Line Fields

The electric and magnetic fields in the sinusoidal steady state of the forward wave in a circular co-axial line of inner and outer radii a and b respectively are, in cylindrical polar coordinates with z along the axis of the line,

$$\begin{aligned}
\mathrm{E}_r &= \mathrm{E}_0 \left(\frac{a}{r}\right) e^{-j\beta z} && \text{(C.14)}\\
\mathrm{E}_\phi &= 0 && \text{(C.15)}\\
\mathrm{E}_z &= 0 && \text{(C.16)}\\
\mathrm{H}_r &= 0 && \text{(C.17)}\\
\mathrm{H}_\phi &= \frac{\mathrm{E}_0}{Z_W} \left(\frac{a}{r}\right) e^{-j\beta z} && \text{(C.18)}\\
\mathrm{H}_z &= 0 && \text{(C.19)}
\end{aligned}$$

where the wave impedance $Z_W = \sqrt{\frac{\mu_0}{\epsilon_r \epsilon_0}}$.

C.4.2 Capacitance and Inductance

The inductance L per unit length and capacitance C per unit length of a dielectric-filled circular coaxial transmission line of inner and outer conductor radii a and b respectively are given by

$$\begin{aligned}
L &= \frac{\mu_0}{2\pi} \log_e \left(\frac{b}{a}\right) && \text{(C.20)}\\
C &= \frac{2\pi \epsilon_r \epsilon_0}{\log_e \left(\frac{b}{a}\right)} && \text{(C.21)}
\end{aligned}$$

C.4.3 Characteristic Impedance

The characteristic impedance a dielectric-filled circular coaxial transmission line of inner and outer conductor radii a and b respectively is given by

$$Z_0 = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_r \epsilon_0}} \log_e \left(\frac{b}{a}\right) \tag{C.22}$$

C.5 Twin Lines

The characteristic impedance of a twin wire transmission line formed from conductors of diameter d separated by a distance s is given approximately by

$$Z_0 = \frac{1}{\pi} \sqrt{\frac{\mu_0}{\varepsilon}} \log \left(\frac{b}{a}\right) \qquad \text{for } s \gg d \tag{C.23}$$

C.6 Poynting Vectors

Neither of the results of the following two sub-sections will be supplied during the examination.

C.6.1 Real Poynting Vector

The real Poynting vector describing the transfer of energy by electromagnetic means is given by

$$\mathsf{N} = \mathsf{E} \times \mathsf{H} \tag{C.24}$$

C.6.2 Complex Poynting Vector

The complex Poynting vector serving a similar purpose is given by

$$\mathbf{S} = \frac{1}{2}\ \mathbf{E} \times\ \mathbf{H}^* \tag{C.25}$$

C.7 Skin effect

Skin depth in a metal at an angular frequency ω is given by

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{C.26}$$

The surface resistivity R_s per square due to skin effect is

$$R_s = \frac{1}{\delta\sigma} = \sqrt{\frac{\omega\mu}{2\sigma}} \tag{C.27}$$

and the wave impedance at the surface is

$$\eta = (1 + j)R_s \tag{C.28}$$

C.8 Waveguide Propagation

1. The dominant mode field configuration for forward wave in rectangular waveguide of interior dimensions a and b along the x and y axes respectively is given by

$$\mathrm{E}_x = 0 \tag{C.29}$$
$$\mathrm{E}_y = -j\eta(\beta_0/\beta_c)\ \mathrm{H}\sin(\pi x/a)e^{-j\beta z} \tag{C.30}$$
$$\mathrm{E}_z = 0 \tag{C.31}$$
$$\mathrm{H}_x = j(\beta/\beta_c)\ \mathrm{H}\sin(\pi x/a)e^{-j\beta z} \tag{C.32}$$
$$\mathrm{H}_y = 0 \tag{C.33}$$
$$\mathrm{H}_z = \mathrm{H}\cos(\pi x/a)e^{-j\beta z} \tag{C.34}$$

2. The propagation constant γ for the either the TE or TM modes of a wave guide satisfies the relation

$$\left(\frac{\omega}{c}\right)^2 + \gamma^2 = \left(\frac{\omega_c}{c}\right)^2 \tag{C.35}$$

3. The cut-off *angular* frequency ω_c of the TE_{1m} or TM_{1m} modes in rectangular waveguide is given by

$$\left(\frac{\omega_c}{c}\right)^2 = \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \tag{C.36}$$

4. The transverse field wave impedances Z_{TE} and Z_{TM} for rectangular waveguide are given by

$$Z_{TE} = (\beta_0/\beta)\eta \tag{C.37}$$
$$Z_{TM} = (\beta/\beta_0)\eta \tag{C.38}$$

5. The group and phase velocities for a wave with angular frequency ω travelling in hollow waveguide in a mode for which the cut-off frequency is ω_c are given by

$$v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}} = (\frac{\beta_0}{\beta})c$$
$$v_g = \frac{\partial\omega}{\partial\beta} = c\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} = (\frac{\beta}{\beta_0})c \tag{C.39}$$
$$\tag{C.40}$$

6. The dimensions of standard X-band waveguide are $a = 22.86$ mm and $b = 10.16$ mm.

C.9 Radiation

C.9.1 Electric and Magnetic Dipole Fields

(a) Electric dipole

In spherical polar co-ordinates at point $P(r, \theta, \phi)$ the non-zero field components of an oscillating small electric dipole of length L carrying a current I and of moment P where $j\omega P = IL$ are

$$E_r = \frac{\beta^2 j\omega P\eta}{4\pi}\left(\frac{2}{(\beta r)^2} - \frac{2j}{(\beta r)^3}\right)e^{-j\beta r}\cos\theta \tag{C.41}$$

$$E_\theta = \frac{\beta^2 j\omega P\eta}{4\pi}\left(\frac{j}{(\beta r)} + \frac{1}{(\beta r)^2} - \frac{j}{(\beta r)^3}\right)e^{-j\beta r}\sin\theta \tag{C.42}$$

$$H_\phi = \frac{\beta^2 j\omega P}{4\pi}\left(\frac{j}{(\beta r)} + \frac{1}{(\beta r)^2}\right)e^{-j\beta r}\sin\theta \tag{C.43}$$

(b) Magnetic dipole

In spherical polar co-ordinates at point $P(r, \theta, \phi)$ the non-zero field components of an oscillating small magnetic dipole of moment $M = \mu_0 IA$ are

$$H_r = \frac{\beta^2 j\omega M}{4\pi\eta}\left(\frac{2}{(\beta r)^2} - \frac{2j}{(\beta r)^3}\right) e^{-j\beta r}\cos\theta \tag{C.44}$$

$$H_\theta = \frac{\beta^2 j\omega M}{4\pi\eta}\left(\frac{j}{(\beta r)} + \frac{1}{(\beta r)^2} - \frac{j}{(\beta r)^3}\right) e^{-j\beta r}\sin\theta \tag{C.45}$$

$$E_\phi = -\frac{\beta^2 j\omega M}{4\pi}\left(\frac{j}{(\beta r)} + \frac{1}{(\beta r)^2}\right) e^{-j\beta r}\sin\theta \tag{C.46}$$

C.9.2 Antenna gains

(b) Small magnetic or electric dipole

The below relation will not be supplied during the examination.

The gain of a small lossless magnetic or electric dipole is 1.5.

(b) Half wave electric dipole

The below relation will not be supplied during the examination.

The gain of a half wave electric dipole is 1.64.

C.9.3 Radiation Resistances

(a) Electric dipole

The radiation resistance of a short electric dipole of length L, operating at a frequency for which the free space propagation constant has magnitude β, is given by

$$R_r = 20(\beta L)^2 \qquad \Omega \tag{C.47}$$

(b) Magnetic dipole

The radiation resistance of a small current loop of radius a, operating at a frequency for which the free space propagation constant has magnitude β, is given by

$$R_r = 20\pi^2(\beta a)^4 \qquad \Omega \tag{C.48}$$

Small loops of other shapes but the same area have the same radiation resistance.

C.9.4 Gain and Effective Area

The below relation will not be supplied during the examination.

The relation between the gain g_t of an antenna in its transmitting role and the effective area of that antenna in its receiving role is

$$A_{er} = \frac{g_r \lambda^2}{4\pi} \tag{C.49}$$

C.10 Lumped Elements

C.10.1 Axial Field of a Circular Coil

In the *magnetostatic approximation*, the field at a point distant z along the axis of a circular coil of radius a is given by

$$H_z(0,0,z) = \frac{Ia^2}{2(a^2+z^2)^{\frac{3}{2}}} \tag{C.50}$$

C.10.2 Inductance Calculations

The self inductance of a single-turn circular coil of diameter D made from wire of diameter d is given, when the currents flow on the surface, by

$$L = \frac{\mu_0 D}{2}\left[\log_e\left(\frac{8D}{d}\right) - 2\right] \tag{C.51}$$

Appendix D

ADVICE ON STUDY FOR EXAMINATIONS

This advice pertains only to study for the June examination on the work of Semester one. Separate advice should be obtained from the lecturer on study for the November examination on the work of semester two.

- Know the SI units for all the electromagnetic quantities.
- Know the content of the Fields Section of the Level 2 Fields and Energy Conversion course.
- Know Maxwell's equations in all of their forms, i.e. differential, integral, time domain, frequency domain, free space, linear media, and non-linear media.
- Know the electromagnetic field boundary conditions in all of their forms.
- Be sure you have a good grasp of a.c. lumped circuit theory.
- Be sure to understand all aspects of the notation for real,time-varying scalar voltages and currents, and real, time-varying vector fields, and of the time-invariant phasors which represent them in the case of sinusoidal steady state time variation.
- Understand the basic properties of simple tuned circuits, including the ideal series, ideal parallel, and practical parallel varieties, and the practical parallel to ideal parallel transformation.
- Be able to derive solutions for forward and backward waves on a uniform transmission line from first principles, and to interpret the solutions in both the time and frequency domains.
- Understand quarter wave transformers, and be able to perform simple design exercises thereon.
- Know the relations between voltage reflection factor and the impedance or admittance on a transmission line, and how each of those quantities transforms along both lossless and lossy lines.
- Be able to interpret standing wave measurements on transmission lines.

- Be familiar with, and competent in performing, all common operations on the Smith Chart.
- Be able to analyse and design single and double stub tuners using transmission lines.
- Study carefully the tutorial problems, the homework questions, and past examination papers, and practice with the remainder of the exercises supplied.

Advice relating to studying for the November examination has not yet been written.

Appendix E

COMMON STUDENT ERRORS

E.1 Objective

This Appendix contains a listing of errors commonly made by students in providing answers to questions in electromagnetic theory. They are provided in the hope that foreknowledge of the common pitfalls will aid in their avoidance.

The list covers observations made at Levels 1 and 3. Little attempt has been made to eliminate duplication, as repetition of an item gives it desirable emphasis.

E.2 Observations at Level 1

E.2.1 Electrical Systems B

(a) November 1997 examination

Question 1

1. All sorts of weird formulae for the field between capacitor plates with dielectric present are presented.
2. It is common, but quite mistaken, to put an r^2 in the denominator of a potential calculation.
3. Sadly, no one seems interested in specifying the direction of the Coulomb force.
4. A small number are still mistakenly placing P in the opposite direction to E in a dielectric.
5. Electric flux is often mistakenly quoted in webers. Aaaaaah!
6. Confusion is found between ϵ_0 and μ_0; the latter is mistakenly used in electrostatics.
7. Misunderstanding of the value of the prefix m for milli is fairly frequent.
8. It seems to be common to calculate the flux emerging from a sphere by mistakenly taking the electric field due to one charge only at one point on the surface and multiplying by the area of that sphere.

9. Students seem to be still in ignorance of the formula for the surface area of a sphere.

10. Electric flux was sometimes mistakenly expressed in Tesla. Gentle reader, it was not good!

Question 2

1. The calculation of field inside a solenoid is frequently badly done.

2. The concept of flux linkage being magnified by the number of turns seems not to have been grasped.

3. Some students have produced unreadable scripts. Is this a good tactic? Definitely not!

4. Quite often students have omitted units from their answers. The examiner wept.

5. Few students can calculate the magnetic field inside a uniform circular cylindrical solenoid. Some are trapped by confusion between the number of turns per unit length and the total number of turns.

6. Very often, current-dependent formulae are mistakenly offered for self inductance or mutual inductance.

7. Students are not strong on dimensional checking. Examples are current equals charge, and magnetic field equals the product of magnetic field with other, dimensioned quantities.

8. Some students are inclined to divide by a vector. Can this possibly mean anything?

9. There is a tendency for some students to add together quantities which are differently dimensioned. This action betrays serious misnderstanding.

Question 3

1. Many students state Faraday's law in terms of an induced current. This is quite wrong.

2. Students are forgetting to put units to their answers.

3. Some students do not know the correct formula for the area of a circle.

4. Some students tudents are mistakenly giving the units of the contour integral of the magnetic field as Am^{-1}.

5. Some students are still identifying the term magnetic field with B .

Global remarks

1. Surface charge density is often quoted in C.

2. There seems to be a mistaken view that when adding scalars one disregards the signs. We see a general mistrust of algebra emerging here.

3. There is a distressing tendency to take the value of an electric field at one point and multiply it by a convenient distance to get the potential difference between two points, one of which is the point at which the field is taken. This is, of course, quite bogus.

(b) Supplementary examination Jauary 1998

Question 1

1. Again the fact that force is a vector seems to go unnoticed.
2. Units are very frequently omitted from answers.
3. Vectors are mistakenly placed equal to scalars.
4. The free space formula is mistakenly used for calculation of the capacitance of a capacitor with a dielectric.
5. The formula $v = Ed$ is misapplied in cases where the electric field is spatially non-uniform.
6. There is very little understanding of the electric flux density generated by a surface charge density on a membrane.
7. The prefixes n and p for nano and pico respectively seem to be not firmly understood.
8. Charges are mistakenly denoted as vector quantities, and multiplied together.
9. I have a suspicion, but only that, that electric flux density means E to some people.
10. The formula for the potential of an isolated charge has been misapplied to the calculation of the potential in a parallel plate capacitor.
11. Quite peculiar formulae for the capacitance between plates have been used.
12. Polarisation is still being put, by some, in the direction opposite to that of the electric field.
13. Units of E are frequently mistakenly given as Cm^{-2}.
14. Units of electric field still being given in NC^{-1}. This is, for engineers, unfortunate.
15. Units of electric flux density are mistakenly given in Wbm^{-2}, and electric field intensity in T.
16. I think I find in some of the responses traces of the notion that electric flux is the integral of E. Can this be a consequence of teaching in high school?
17. Units are sometimes completely omitted. Can this be a behaviour which is tacitly encouraged in high school?
18. Units of magnetic field have mistakenly been given as volts.

Question 3

1. Discussion of Faraday's law is sometimes conducted in terms of induced currents not induced voltages. This is a serious misunderstanding of the law. Could it have its origin in high school?

2. The magnetic field inside a toroid is not well known.

3. The volume current density is often confused with the displacement current density.

4. The flux distribution in a toroid is often seriously misunderstood.

5. Pronouns without antecedents still occur fairly frequently.

6. The role of a number of turns in a toroid in influencing the core magnetic field is not understood.

7. Students are confusing the concepts of a "line integral around" and "enclosed by" and are using the latter phrase when the former would be needed, and using the phrase "through a contour" which has no clear meaning.

8. In Ampere's law as modified by Maxwell, B is sometimes mistakenly written for E.

9. I think there is a widespread misunderstanding of the shape of the magnetic field of a toroid.

10. Students use the phrase "flux around a contour" instead of "flux linked by a contour". I am not sure of what they have in mind.

E.3 Observations at Level 3

E.3.1 Fields Lines and Guides

(a) June 1997 examination

1. Students lack the ability to reproduce or construct a solution for a capacitor discharging into a resistor.

2. Students are confused about reflection factor and transmission factor relations in transmission lines.

3. There is sometimes a lack of knowledge of propagation velocity in transmission line.

4. Some students are at sea in recognising small bits of transmission line as equivalent to lumped circuit elements.

5. Some students lack knowledge about the velocity in a transmission line and how a dielectric affects that propagation velocity.

6. Oscillations are mistakenly seen as originating from a lumped passive RC circuit.

7. Complex numbers are mistakenly given in answers to questions about time-varying real quantities.

8. There is an inability to translate phasors to time variables and vice versa.

9. The expression for η is sometimes used to calculate the velocity of light. Is this a complete wipeout of Level 1, Level 2 and Level 3 electromagnetic teaching?

10. There is a widespread mistaken belief that in a transmission line, the transmission factor = 1 - reflection factor, meaning that the basic concepts of travelling waves on transmission lines are completely misunderstood.

11. The input signal at the source end of a transmission line is sometimes mistakenly seen as reaching the load instantaneously.

12. There is a widespread mistaken belief that a short length of open circuited transmission line is equivalent to a large capacitance.

13. Factors of $\sqrt{3}$ are sometimes mistakenly introduced into peak value phasors where even $\sqrt{2}$ would not have been appropriate.

14. The direction in the rubric to state units in answers was frequently ignored.

15. Phasors are sometimes mistakenly made functions of time, and real variables are mistakenly expressed as complex numbers.

16. Written explanations are sometimes difficult to follow as a result of the use of pronouns without properly defined antecedents.

17. Charge density per unit area is mistakenly expressed in Coulombs.

18. Misunderstanding of the term "peak value, i.e. not r.m.s. phasors" has occurred.

19. There is a lack of knowledge of the value of ϵ_0.

20. There is cheerful but mistaken acceptance of values of voltage reflection factor greater than unity in lines with passive terminations.

21. Inductance is mistakenly being quoted in Farads; capacitance is mistakenly being quoted in Coulombs; magnetic field H is being mistakenly quoted in Wb/m and in T/m; power density of a uniform plane wave is being mistakenly quoted in Wm^{-1}; the Poynting vector is mistakenly given as the product of D and E; magnetic flux density B is mistakenly given as $\mu_0\mathsf{E}$; magnetic field H is mistakenly given as E/μ_0; and units of time are mistakenly given as F.

22. The usual confusion between ω and f occurs.

23. Mistakenly adding differently dimensioned quantities in an expression, e.g. R + LC; R + L + C has been noticed more that once.

24. The formula for capacitance of a parallel plate capacitor was mistakenly given as $\epsilon_0 A/d^2$.

25. Mistakenly giving dimensions to the ratio of identically named and identically dimensioned quantities.

26. Frequent mistaken attempts to used differential forms of Maxwell's equations at a discontinuity between media occur.

(b) September 1997 examination

Question 3

1. Some students are unable to recognise the consequences of wave travelling in -z direction.

2. Some have not noticed that antenna admittance rather than impedance was given.

3. Not knowing the formula for Poynting vector.

4. Not knowing the units of electric field.

5. Incorrect normalisation of load admittance to a 50 Ω line.

6. Inability to calculate electromagnetic wave length for a given frequency.

7. Students do not have a clear understanding of the relation between the descriptions of a uniform plane wave in the time domain and the frequency domain.

8. Frequency is mistakenly quoted in m^2/s.

9. There is a lack of awareness that a wave travelling in the -z direction would carry power in that direction.

10. Scrambling of the relation $\omega = c\beta$ occurs.

(c) November 1997 examination

Question 1

1. Some students have produced a completely garbled version of Maxwell's equations.

2. Some students seem mistakenly to be able to divide by a vector.

3. Dividing vectors (each being a curl) and cancelling the del cross symbols, i.e. pure mathematical gibberish, mistakenly occurs.

Question 2

1. Some students have no concept of the dimensionality of things - quantities are mistakenly assembled in dimensionally inappropriate combinations.

2. Tutorials seem to have been skipped.

3. The concept of transmission loss is seriously misunderstood - often the difference between transmitted and received power is taken. This illustrates a failure to grasp a concept, and a failure to think as an engineer.

4. Decibels are almost universally but mistakenly being calculated as 20 $\log_{10}$ of a power ratio. The represents a failure to understand the decibel concept.

5. Some students cheerfully but mistakenly take the logarithm of a dimensioned quantity.

6. Maybe we need a slow lecture on all aspects of dB.

7. A matched antenna is assumed to be one which resembles another - not one which has the optimum termination for power transfer.

8. Very few students can make a coherent and correct statement of the definition of effective area.

9. There is a cheerful but mistaken acceptance of received power greater than transmitted power in a radiation situation.

10. Effective area is mistakenly said to be the area of transmitted power. I don't see how you can make sense of this.

Question 3

1. In waveguide propagation, the wave and group velocities are often mistakenly considered to be just the free space velocity.

2. The wave impedance in a waveguide is often mistakenly considered to be the free space characteristic impedance.

3. The translation of α to dB/m is widely misunderstood; many mistaken attempts to take the log of α were made.

4. There is widespread misunderstanding of the concepts of β, β_0, λ_g, and λ_0.

5. Quite often $\frac{1}{2}\mathbf{E} \cdot \mathbf{E}^*$ is taken as power flowing down a waveguide. There are many mistakes here. How many can you count?

6. Power is mistakenly seen as V$/Z^2$.

7. There is an almost universal heresy that the characteristic impedance of the TE and TM propagating modes of a waveguide are all equal to the characteristic impedance of free space.

8. TEM waves and uniform plane waves are mistakenly equated.

Global remarks

1. The surface area of a sphere is sometimes mistakenly seen as πr^2.
2. Students are still omitting units from their answers.
3. The properties of dispersive propagation seem not to be understood.

(d) June 1998 Examination

Question One

1. Very often students make the mistake of calculating transmission line parameters of characteristic impedance and propagation velocity using the total inductance of the line rather than those parameters per unit length.
2. Some students seem to mistakenly interpret the word schematic circuit diagram to be restricted to a diagram containing lumped circuit elements only.
3. A small number of students are still having things happen instantaneously at the load of a transmission line problem where the excitation is at the source. Should they be whipped ?
4. It also seems that some students do not know the formula for propagation velocity of a transmission line in terms of the inductance per unit length and capacitance per unit length. In the past such students would be whipped.
5. The number of students who are calculating the velocity on a transmission line by mistakenly using the total inductance and total capacitance rather than those parameters per unit length is close to a majority.

Question Two

1. There is still a significant number of students are still omitting units from their answers and are, as promised, losing a significant number of marks.
2. Some students believe that to invert a complex number you separately invert the real and imaginary parts. Where this nonsense comes from I have no idea.
3. Some students are offering a function of time as an answer to a question asking for a phasor to be created. Apparently such students have not noticed my screaming that this should never be done.
4. It would seem that many students have not detected the fact that knowing the must knows as listed in the experimental electrical engineering three notes would be of great benefit in performing any level three examination.
5. Many students make a mess of the question on designing a quarter wave transmission line transformer. Some do not see how simple the question is, and supply elaborate and frequently incorrect formulae for its solution. Many are totally mistaken about the role the dielectric permittivity plays in devising the solution. The conclusion one can draw is that students should practice simple quarter wave transformer design.

6. One student has not attempted the radiation question at all. This seems to be highly regrettable.

7. Some students cannot draw a single stub tuner. That puts them rather behind the eight ball in the task of designing one.

8. There are still students who offer a complex number as an answer to a question about a real quantity. Should such students should be flailed ?

9. Some students are under the mistaken impression that the negative susceptances are in the upper half of the admittance Smith Chart. Very sad.

10. Some students are giving a result in volts for a question which asks for a current. Some people seem to be using the symbol I for units of current. A strange one. Surely this is very strange behaviour.

11. One student appears to be unfazed by giving a quarter wave transformer length of 2.5 nano metres in a problem for which the frequency is 100MHz. Surely the incongruity should prompt some comment, or preferably some amendment of the result.

Question Three

1. Many students are still incorrectly calculating decibels by applying a factor 20 the log to base 10 of a power ratio.

2. Some people are inclined to quote the units of electric fields as volts. This is very sad.

3. Some students are still insisting that the transmission loss between two antennas is related to the difference between the transmitted power and the received power.

4. In the answers to question three, dimensionally incorrect equations abound.

5. One student offered Maxwell's equations because he thought they should have been asked for in the examination and hoped to get marks for them. Unfortunately his version of the equations was seriously flawed.

6. I see continuing evidence that some students do not know the formulae for the area of a sphere.

7. Students are still inclined to attach significance to the logarithm of a dimensioned quantity.

8. It may be noticed that some students, when given a radius in a question and having applied that in a formula, may also use the same figure when a diameter is required.

9. I have a suspicion that some students did not study the radiation chapter at all.

10. More often than not, a dimensionally incorrect formula is offered for the quality factor of a tuned circuit.

11. Very often an incorrect formula is used for the relationship between frequency, wavelength and wave velocity.

12. Occasionally a negative value for the gain of an lossless antenna is given. How can this be?

13. On some occasions the value for ϵ_0 has been used instead of μ_0.

14. Sometimes electric field is mistakenly seen to be linearly proportional to power density.

(e) November 1999 Examination

1. We will have to buy several more green pens.

2. I am again astounded at the variety of incorrect responses to the early parts of question 1. Almost no students can correctly translate a phaser into an appropriate real function of time. The answers are mostly alphabet soup, that is inappropriate combinations of symbols which are sometimes meaningless in the time domain, and sometimes utterly confound the most elementary expectation that a sign wave will result.

3. What can we do with such students? Surely letting them float through on a sea of such basic ignorance is inappropriate.

4. Very few students seem to be able to divine the fact that for a simple travelling wave on a lossless transition line travelling in a particular direction, if you move your point of observation one quarter of a wave length in that direction, the amplitude is preserved and the phase is retarded by 90 degrees. Neither of these facts seems to be firmly grasped (elementary).

5. Perhaps the most distressing feature of these students answers is a tendency to offer a complex number as an answer to a question where a real function has been sort. There are some students who do not even attempt to convert a phaser into a time function nor offer a view of what happens to a phaser representing a travelling wave when you move the point of observation further down the line. There is a large number of students who when asked for the impedance of a circuit at its half power points gives a real number (admittance) (not impedance). Perhaps they have in their mind the notion that impedance implies magnitude of admittance and cannot be complex. If this be so it is difficult to imagine where that notion can have come from. I have thrown away scrap paper enclosed with some examination papers as it is time consuming to handle it.

6. It appears that one student is using the symbol A for the units of admittance an enterprising effort.

7. There is still a determined band of students who insist that a parallel resonant circuit has an impedance minimum at resonance. We can hope that they will eventually die. We can at least hope that they will not propagate.

8. There is a small number of students who insist that the correct units for something with the symbol V would be metres per second. This is despite the fact that it has been defined as a voltage.

9. One student is giving henries for the units of admittance. I think he has strayed some way from the correct path.

10. Mr. Heath Justin Stephens has a note in his paper that he had car trouble of two kinds and also left his calculator in the car. He indicates he cannot really concentrate with which I can sympathise. He has other comments that I could not really make sense of.

11. Students are still prone to give dimensioned arguments to the sine, cosine, exponential function, etc. How you can attach a meaning to this escapes me.

12. Many students have mistakenly identified the phase change of a forward wave with the phase change of a reflection factor and have given an answer for the latter instead of the former.

13. All these remarks and those which follow until notified otherwise will be related to question 1.

14. One student confesses that he cannot remember how to multiply two complex numbers.

15. Now we proceed to discuss question two.

16. Most students have not grasped the idea of considering only the surface skin of a round conductor and calculating the resistance of that conductor. They do not understand that you could unpeal such a surface skin.

17. Many students are interpreting surface resistivity whose units are said to be ohms per square as something which has a resistance per square metre. Clearly a much bigger song and dance will have to be made about the slightly inappropriate name, and a translation of the name to fully correct terminology will have to be made.

18. Some students in writing Maxwell's equations are prone to equate the phaser variables with the sinusoidal steady state (real) variables. There is considerable confusion shown by doing this.

19. One student is cheerfully adding the parameters ** 0 and absilom 0 to get what I cannot imagine.

20. I could make with chalk and blotting paper a convincing illustration of the basic facts of resistance of a conductor in the skin depth situation, but how would this benefit students who do not come to lectures? The matter is currently widely misunderstood and would remain widely misunderstood. There is the very common error that the resistance of the conductor is seen as the surface resistivity times the area of the conductor. Clearly if there is a recitation of correct answers in the paper there will have to be considerable ** time spent on this matter.

21. Further comments are now about question 3.

22. It is common for students to mistakenly omit the units from the propagation constant beta. This occurs even when students are diligent about inserting units for most of the quantities. We might have the examiners meeting give a special pleading for Mr. Stephens, Heath Justin who appears to have had terrible experiences on the way to the examination.

23. We see continuing evidence of the confusion between frequency and angular frequency and in particular using the value of one where the value of the other is required.

24. Sometimes the units of a propagation constant are given as amps per metre. This is highly imaginative to say the least.

(f) Supplementary Examination November 1999 - Paper 2

Question 1

1. It is frequently very difficult to read what students have presented in their answers. No notice appears to have been taken to the directive to express answers to questions clearly and to write them legibly in the rubric to the paper.

2. There are still students who are inclined to provide the symbol D and units of Coulumns per metre squared for the concept of magnetic field.

3. Although some students realise that a factor of two pi is involved in converting frequency and hertz to angular frequency, they have unfortunately placed it in the denominator, rather than in the numerator where it belongs.

4. A large number of students are still getting units of watts for the concept of power density per unit area.

5. The question on boundary conditions involving state of dielectric material is often mistakenly seen as a question on a parallel plate capacitor and bogus reasoning based on that mistaken view is advanced.

6. The properties of a basic simple series tuned circuit are very rarely understood.

7. Quite often a value of frequency is used in the formula where angular frequency would have been required.

8. Students seem to confuse volume current density with surface current density and seek a formula for the former when a question is asked about the latter.

Question 2

1. Practically no one understands the concept of surface resistivity and how it may be used to calculate the resistance of a conductor on which only skin currents are flowing. Almost always the cross sectional area of the conductor appears in the calculation. This suggests that the concept of skin currents is entirely misunderstood.

2. It appears that some students are in great difficulty when asked to recite Maxwell's equations. (reproduce)

3. One student shows that he misunderstands the term cylinder by equating it with the concept of a right circular cylinder.

Appendix F

HOMEWORK

F.1 Homework 1

Distributed:

Hand in:

1. Construct phasors to represent:

   (a) $v(t) = 10\cos(\omega t + \pi/2)$ V; and

   (b) $i(t) = 5\sin\omega t$ A.

2. Find the real voltage as a function of position and time represented by the phasor $v(z) = \mathrm{V}_f e^{-j\beta z}$ where $\mathrm{V}_f = (3 + 3j)$ V.

3. A d.c. voltage source of internal voltage 1200 volts and internal resistance 10 Ω is connected at $t = 0$ to 10 metres of lossless co-axial transmission line. Measurements at low frequencies show that the length of line has a total capacitance of 1.0 nF and a total inductance of 2.5 μH. The line is teminated in a load resistance of 30 Ω.

   (a) Determine the characteristic impedance and the wave velocity for the coaxial line.

   (b) Draw a schematic circuit diagram and below it a lattice diagram of the forward and reverse waves of voltage.

   (c) Determine the transit time for a wave front to travel from the source to load and the reflection coefficients for each end of the line, and show them on the lattice diagram.

   (d) Show details of the voltage distribution as a function of position along the line at 1.5 and 2.5 times the one-way transit time.

   (e) Show details of the voltage waveform as a funtion of time at the load terminals.

   (f) What is the steady state voltage distribution for this line with the source and load specified?

F.2 Homework 2

Distributed:

Hand in:

1. Calculate the inductance per unit length of an air-cored co-axial line of radii a (inner conductor) and b (outer conductor).

2. What is the input impedance of each of the lossless lines shown in Figure F.1?

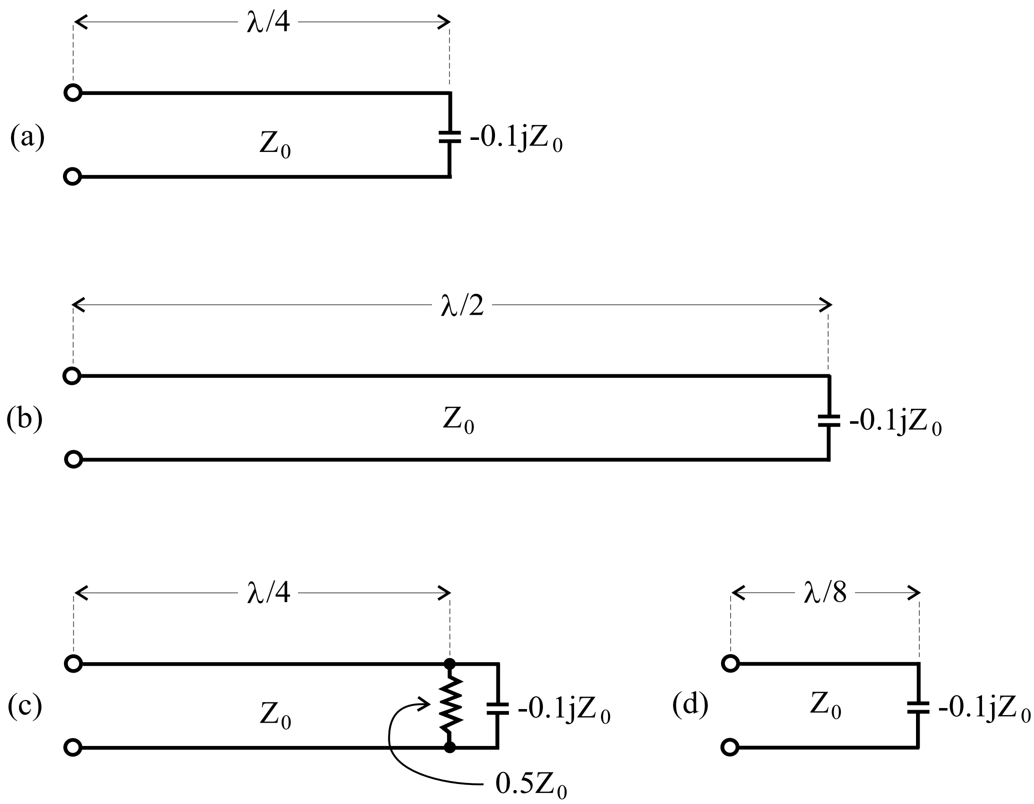


Figure F.1: Lossless transmission lines.

3. A transmission line has the following distributed parameters per unit length:

   $L = 0.5\ \mu$H/m, $R = 2.0\ \Omega$/m, $C = 50$ pF/m, and $G = 0$ S/m.

   Calculate at a frequency of 31.8 MHz the characteristic impedance, and the attenuation and phase constants.

4. A *distortionless line* is defined as one in which α is independent of frequency and β is proportional to frequency. Show that this condition occurs when $R/L = G/C$.

5. Calculate the input impedance of 1000 m of the transmission line described in Problem 3. Explain why this does not depend on the load impedance Z_L.

6. An impedance of $(100 + j100)\ \Omega$ is placed as a load on a lossless transmission line of characteristic impedance 50 Ω. Find the reflection coefficient in magnitude and phase at the load end. What is its magnitude and phase at the input end if the line is $3\lambda/8$ long at the operating frequency? What happens as the frequency varies?

F.3 Homework 3

Distributed:

Hand in:

1. Calculate the attenuation constant α in nepers per metre, and the transmission loss in dB per metre, for an air-filled co-axial line of inner and outer conductor radii a and b, respectively, in terms of conductor material conductivity σ and the proportions of the line.
2. How does this result scale with frequency?
3. What are the losses for an air-filled copper co-axial line of inner conductor radius 2 mm and outer conductor radius 7 mm at frequencies of:
   (a) 100 kHz; and
   (b) 10 Mhz?
4. What length of this cable would you use as a high power dummy load to achieve, at a frequency of 1.0 GHz, and for any value of load impedance, an input VSWR of less than or equal to 1.5?

F.4 Homework 4

Distributed:

Hand in:

1. A rectangular waveguide of dimensions 22.86 mm x 10.16 mm is operating in the dominant TE_{10} mode at a frequency of 10 GHz. Determine for this mode:
   (a) the cut-off frequency f_c,
   (b) the phase constant β,
   (c) the wavelength λ_g,
   (d) the phase velocity v_p,
   (e) the group velocity v_g,
   (f) the wave impedance Z_{TE}.
2. If the peak amplitude of the electric field in the waveguide above is 1.0 kV/m, calculate the power carried by the guide.

3. For the waveguide described above, calculate the attenuation constant in dB/m of the dominant mode at the frequency of 5 GHz, at which frequency that mode is below cut-off.

Appendix G

TUTORIALS

G.1 Tutorial 1

1. For the coaxial transmission line shown in Figure G.1, in which the centre conductor has dc current i, and steady charge q per unit length, use the integral forms of Gauss' and Ampere's laws to calculate the values of the electric field and the magnetic field at a radius r. Assume that the centre conductor is supported by a non-magnetic dielectric of dielectric permittivity ϵ.

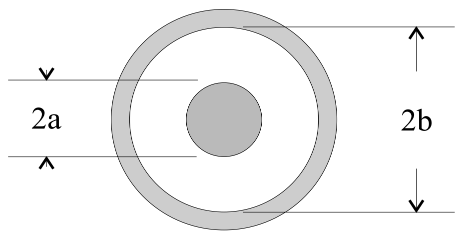


Figure G.1: Co-axial line configuration.

2. A lossless transmission line, initially charged to a dc voltage V, is shorted at its input at $t = 0$. Sketch the voltage $v(t)$ at the open-circuit output end.

3. A pulse generator consists of 10 metres of 50 Ω transmission line in which the velocity is 2×10^8 m/s, charged to 6 kV dc, and connected to a load impedance of 25 Ω by a relay which is closed at time $t = 0$. Sketch to definite scales the output pulse obtained. Calculate the peak output power and the total output energy in the pulse.

G.2 Tutorial 2

1. In the circuit in Figure G.2, the transmission line has air dielectric, and the resistors are lumped elements.

   (a) Sketch the frequency response of the circuit over the frequency range from 0 to 500 MHz.

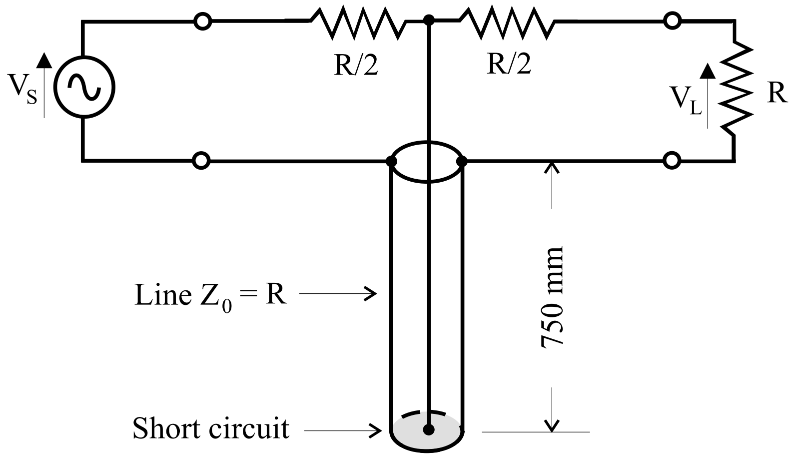


Figure G.2: Transmission line in a lumped parameter circuit.

(b) Sketch the frequency response when the short circuit on the transmission line is replaced by an open circuit.

2. (a) At a frequency of 1000 MHz, to what capacitance is 10 cm of 50 Ω line, shorted at its end, equivalent? Assume that the dielectric is air.
   (b) If the line has instead a dielectric with dielectric constant 2.25, what is the capacitance?

3. What is the input impedance at a single frequency for the transmission lines shown in Figure G.3, expressed as a large or small inductance or capacitance? In Figure G.3, δ is a length very much less than the wave length λ on the line at the frequency of interest.

4. What current flows in the short circuit at the end of the lossless line shown in Figure G.4?

5. If a coaxial cable with loss 0.2 dB per metre is used as in Figure G.5, what is the VSWR:

   (a) at the load end; and
   (b) at the source?

G.3 Tutorial 3

1. A 75 Ω line is terminated in $(50 + j100)$ Ω.

   (a) What is the VSWR? What percentage of forward power is reflected from the load?

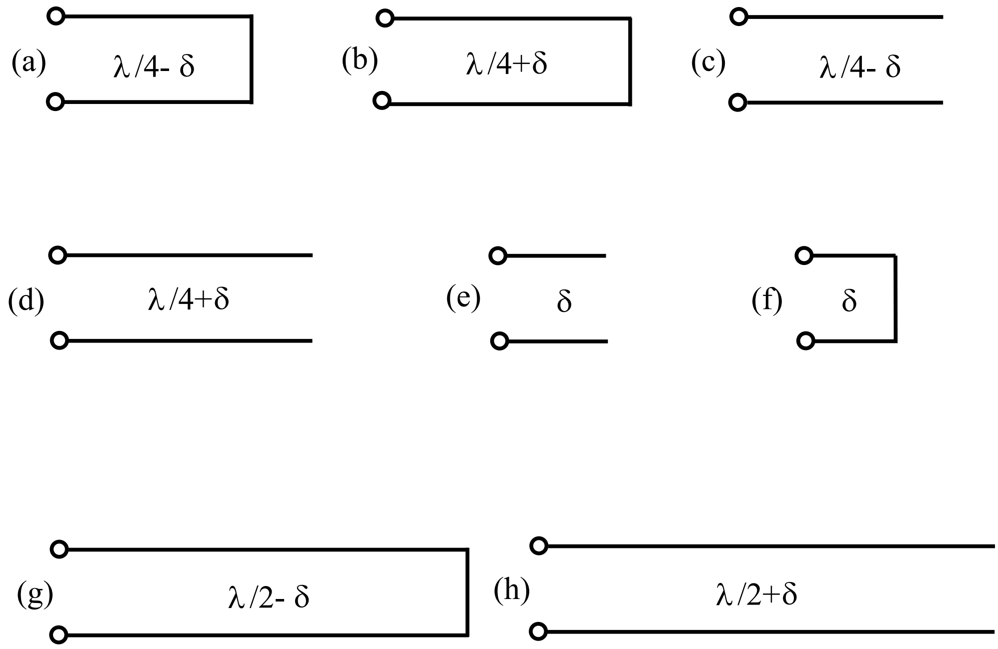


Figure G.3: Various open and short circuit transmisison lines.

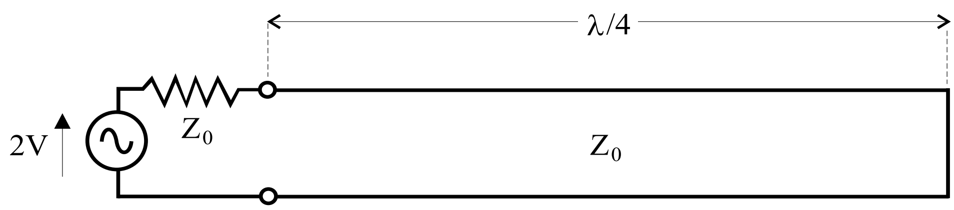


Figure G.4: Shorted lossless transmision line with matched source.

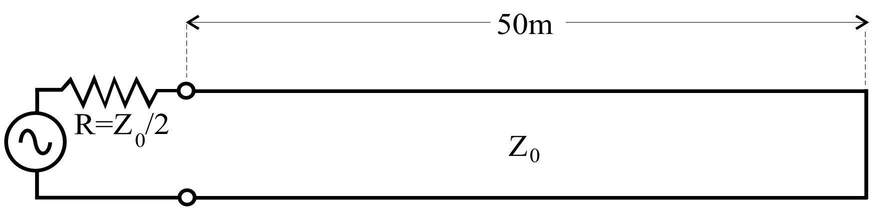


Figure G.5: Shorted lossy transmision line with unmatched source.

(b) What is the distance from the load back to the first voltage minimum, and to the first voltage maximum?

(c) If the line is to be matched by a single stub, what is its length and position? Both the line and the stub have characteristic impedance $Z_0 = 75\ \Omega$.

2. In the matching system shown in Figure G.6, l_1 and l_2 are line lengths and s_1 and s_2 are the lengths of the stubs.

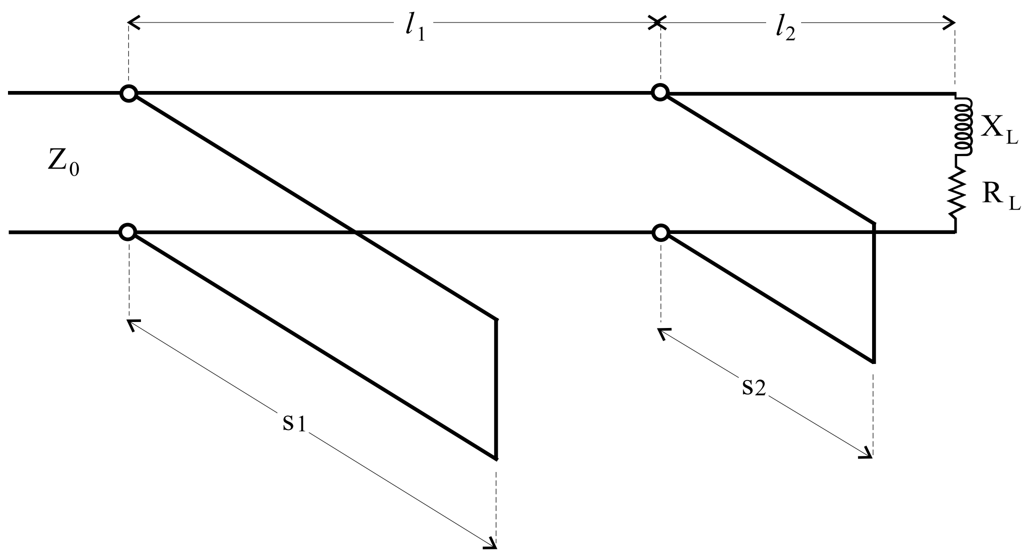


Figure G.6: A double stub matching system.

(a) If $l_2 = 0$ and $l_1 = 3\lambda/8$, what values of R_L cannot be matched?

(b) If $l_2 = 0$ and $l_1 = \lambda/4$, what values of R_L cannot be matched?

(c) If $l_2 = \lambda/10$ and $l_1 = 3\lambda/8$, and $R_L = Z_0/2$ and $X_L = Z_0$, what stub lengths s_1 and s_2 are required for a match?

(d) What is then the VSWR in line l_2, and what is it in line l_1?

Appendix H

HOMEWORK ANSWERS

In this Appendix, the original questions are reproduced along with the answers. The section numbering used in this Appendix follows that used in Appendix F wherein the questions originally appeared.

H.1 Homework 1

1. Construct phasors to represent:

   (a) $v(t) = 10\cos(\omega t + \pi/2)$ V
   Answer

$$\begin{aligned} \mathrm{V} &= 10e^{j\pi/2} \text{ V} \\ &= 10j \text{ V} \\ &= 10\angle 90^\circ \text{ V} \end{aligned}$$

   are suitable alternative forms.

   (b) $i(t) = 5\sin\omega t$ A.
   Answer
   We must first convert to a cosine function, using the relation $\sin(\omega t) = \cos(\omega t - \pi/2)$. Note that the relation $\sin(\omega t) = \cos(\pi/2 - \omega t)$ is not suitable because it reverses the sign of ωt. Thus $i(t) = 5\cos(\omega t - \pi/2)$ A. The phasor is then

$$\begin{aligned} \mathrm{I} &= 5e^{-j\pi/2} \text{ A} \\ &= -5j \text{ A} \\ &= 5\angle -90^\circ \text{ A} \end{aligned}$$

2. Find the real voltage as a function of position and time represented by the phasor $v(z) = \mathrm{V}_f e^{-j\beta z}$ where $\mathrm{V}_f = (3+3j)$ V.

   Answer

It is first convenient to convert V_f to polar form, ie $V_f = 3\sqrt{2}e^{j\pi/4}$ V. Then $V(z) = 3\sqrt{2}e^{-j\beta z + j\pi/4}$ V. Inserting a factor $e^{j\omega t}$ and taking the real part, we obtain the real voltage function

$$v(z,t) = 3\sqrt{2}\cos(\omega t - \beta z + \pi/4) \text{ V} \tag{H.1}$$

3. A d.c. voltage source of internal voltage 1200 volts and internal resistance 10 Ω is connected at $t = 0$ to 10 metres of lossless co-axial transmission line. Measurements at low frequencies show that the length of line has a total capacitance of 1.0 nF and a total inductance of 2.5 μH. The line is terminated in load resistance of 30 Ω.

   (a) Determine the characteristic impedance and the wave velocity for the coaxial line.

   Answer

   The inductance and capacitance per unit length are 250 nH/m and 100 pF/m. Hence

$$\begin{aligned} Z_0 &= \sqrt{\tfrac{L}{C}} &&= 50\,\Omega \text{ , and} \\ c &= \tfrac{1}{\sqrt{LC}} &&= 200 \text{ Mm/s} \end{aligned}$$

   (b) Draw a schematic circuit diagram and below it a lattice diagram of the forward and reverse waves of voltage.

   Answer

   See Figure H.1.

   (c) Determine the transit time for a wave front to travel from the source to load and the reflection coefficients for each end of the line, and show them on the lattice diagram.

   Answer

   Transit time $T = \dfrac{10 \text{ m}}{200 \text{ Mm/s}} = 50$ ns

$$\begin{aligned} \Gamma_L &= \frac{30-50}{30+50} = -\frac{1}{4} \\ \Gamma_S &= \frac{10-50}{10+50} = -\frac{2}{3} \end{aligned}$$

   These values are shown on the lattice diagram of Figure H.1.

   (d) Show details of the voltage distribution as a function of position along the line at 1.5 and 2.5 times the one way transit time.

   Answer

   At 1.5 times the transit time we have a forward wave of 1000 V has reached the load and a reflected wave of -250 V has reached half way back to the source. At 2.5 times the transit time we have the -250 V has reached the source and produced and additional forward wave of 167V, which has travelled half way to the load. Hence the diagrams shown in Figure H.2.

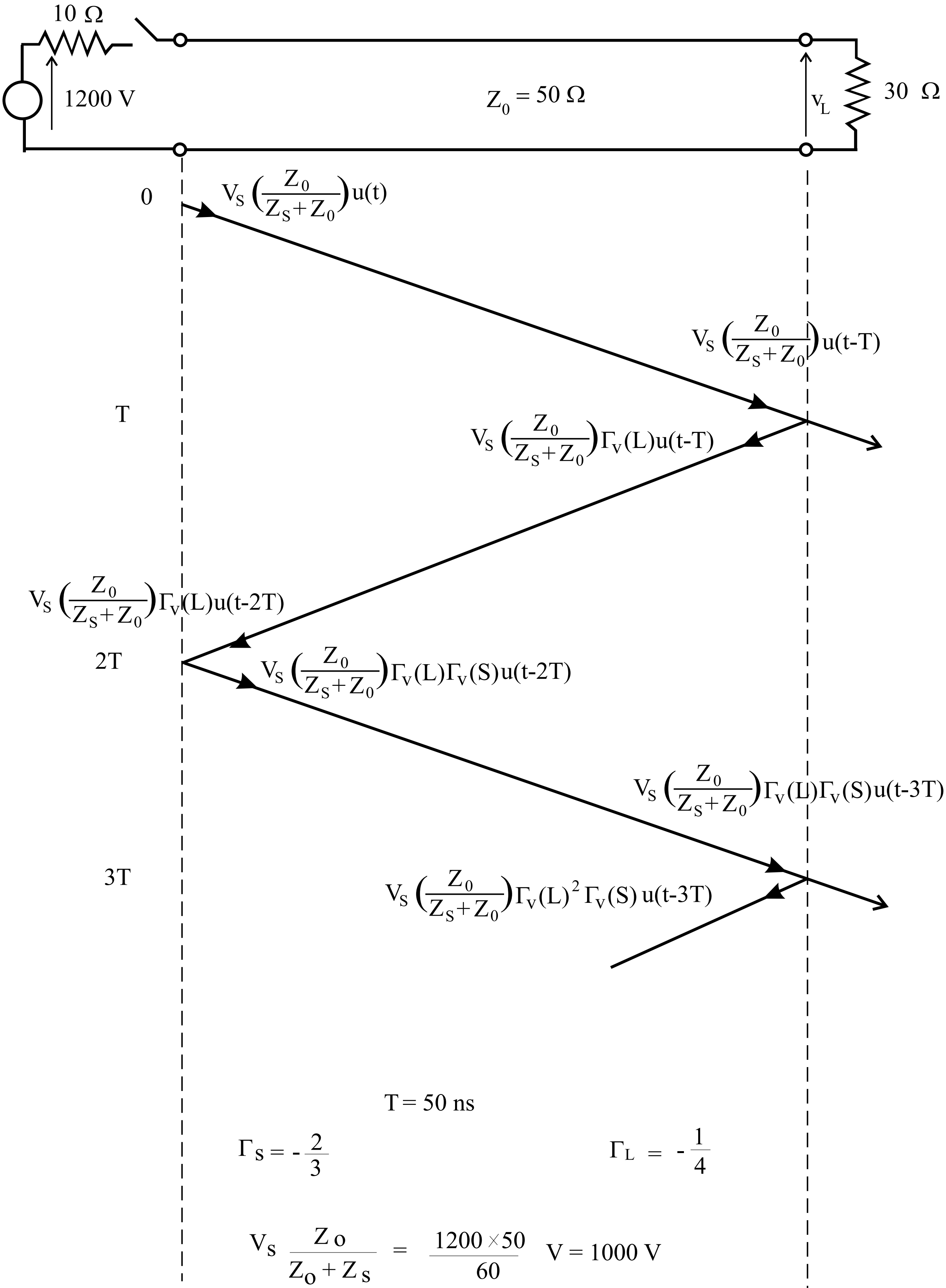


Figure H.1: Lattice diagram for the coaxial transmission line.

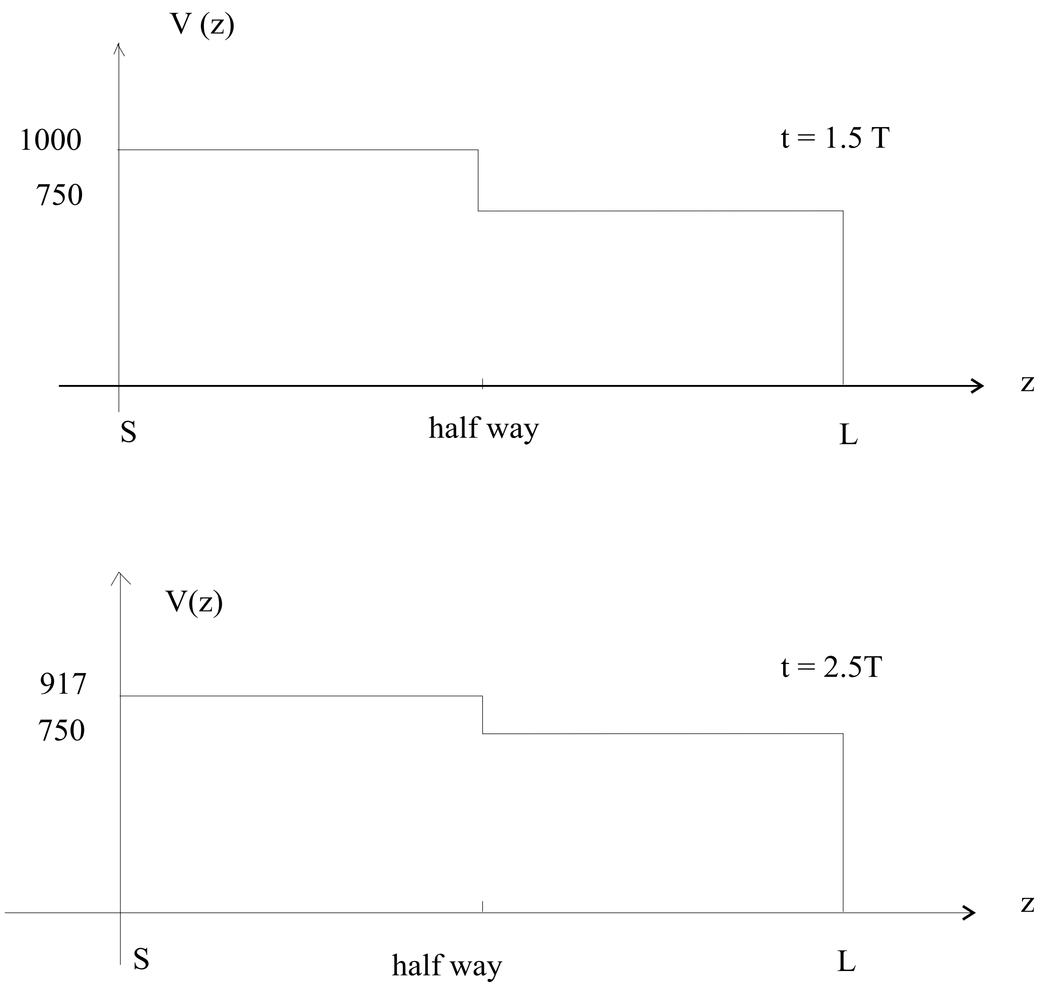


Figure H.2: Total voltage at times 1.5T and 2.5T

(e) Show details of the voltage waveform as a fuction of time at the load terminals.

Answer

At the load terminals we have a voltage

$$\begin{aligned} & V_S \frac{Z_0}{Z_S + Z_0}(1+\Gamma_L)u(t-T) = 1000V\left(\frac{3}{4}\right)u(t-T) \\ + \quad & V_S \frac{Z_0}{Z_S + Z_L}(1+\Gamma_L)\Gamma_S\Gamma_L u(t-3T) = 1000V\frac{3}{4}\frac{1}{6}u(t-3T) \\ + \quad & \text{more terms with an extra factor of } \frac{1}{6} \text{ each time.} \end{aligned}$$

A diagram of the waveform appears in Figure H.3.

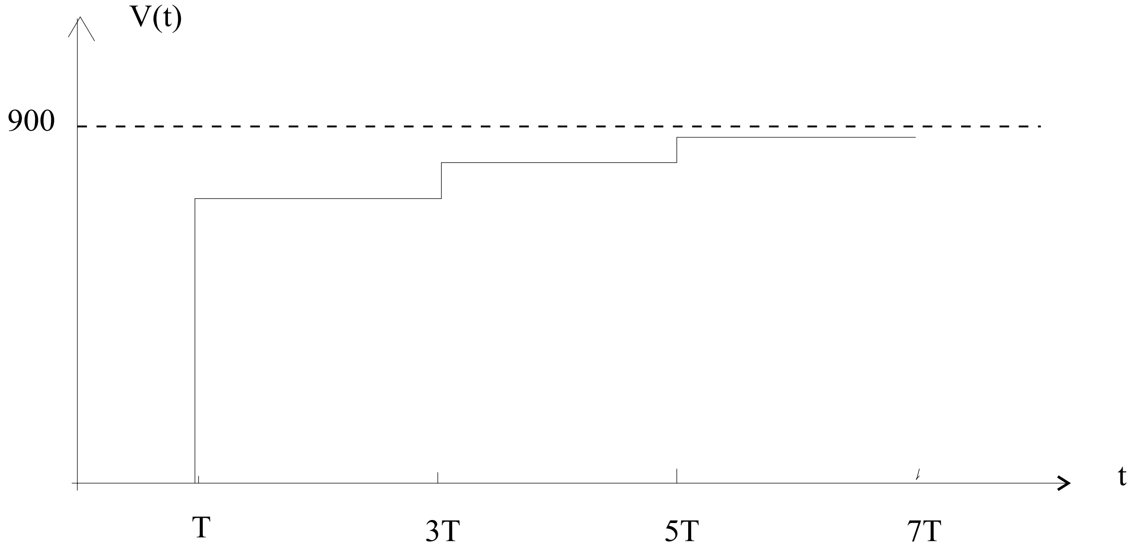


Figure H.3: Load end voltage as a function of time.

(f) What is the steady state voltage distribution for this line with the source and load specified?

Answer

Eventually the line takes a steady state voltage of $V_S \frac{Z_L}{Z_S + Z_L} = 1200\frac{30}{10+30} = 900V$.

H.2 Homework 2

1. Calculate the inductance per unit length of an air-cored co-axial line of radii a (inner conductor) and b (outer conductor).

Answer

In respect of this question, the student could assume a short length δz of line, a line charge density q per unit length on the inner conductor, and a current i on that conductor. Then by employing Gauss' law of electrostatics, an expression for the radial component of electric flux density, and then the electric field, at a distance r from the centre can be derived. By employing Ampere's law of magnetostatics, an expression for the circumferential component of the magnetic field, and then the magnetic flux density, at a distance r from the center can be derived.

Integration of the electric field expression, with due care to signs and the order of limits of the integral, should then give the potential of the inner conductor relative to the outer conductor, and should lead, after this result is divided into the charge per unit length, to an expression for the capacitance per unit length.

Calculation of the flux linked by a surface of length δz and width from the inner to outer conductor, and dividing by the current i, should give the inductance of the line. The results should be

$$\begin{aligned} C &= \frac{2\pi\varepsilon}{\log_e\left(\frac{b}{a}\right)} \\ L &= \frac{\mu_0}{2\pi}\log_e\left(\frac{b}{a}\right) \end{aligned}$$

2. What is the input impedance of each of the lossless lines shown in Figure H.4?

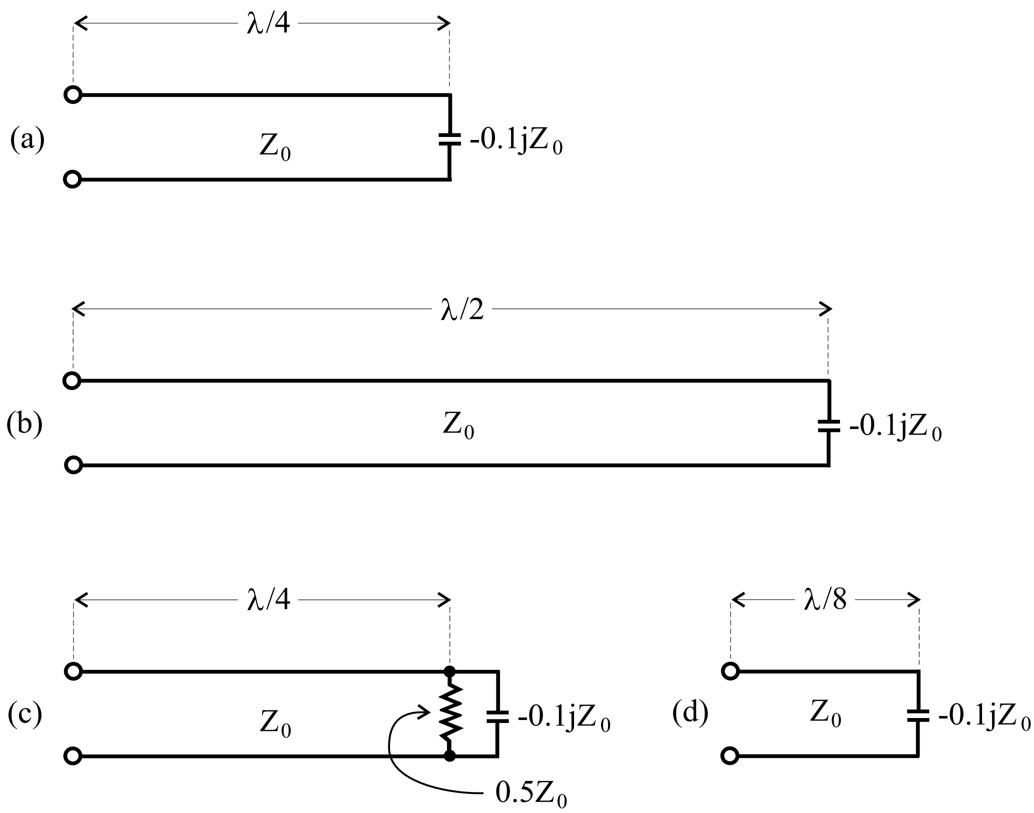


Figure H.4: Lossless transmission lines.

Answer

The answers to the first three parts of this question may be easily obtained from the facts that a quarter wave length of line transforms a load impedance Z_L into an input impedance of $Z_I = Z_0^2/Z_L$, while a half wave length of line transforms a load impedance Z_L into itself. The answer to the last question is obtained by using the equation

$$\frac{Z_I}{Z_0} = \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} \tag{H.2}$$

and noting that both $\cos\beta l$ and $\sin\beta l$ are equal to $1/\sqrt{2}$. The results are:

(a) $Z_i = 10jZ_0$;

(b) $Z_i = -0.1jZ_0$;

(c) $Z_i = (2 + j10)Z_0$; and

(d) $Z_i = 0.818jZ_0$.

3. A transmission line has the following distributed parameters per unit length:

$L = 0.5\ \mu\text{H/m}$, $R = 2.0\ \Omega/\text{m}$, $C = 50\ \text{pF/m}$, and $G = 0\ \text{S/m}$.

Calculate at a frequency of 31.8 MHz the characteristic impedance, and the attenuation and phase constants.

Answer

At 31.8 Mhz, $\omega L = 100\Omega/\text{m}$ and $\omega C = 0.01\text{S/m}$. Hence the characteristic impedance is

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{2 + j100}{j0.01}} = 100\sqrt{1 - 0.02j} \tag{H.3}$$

Using a two-term binomial expansion we obtain

$$Z_0 = (100 - j1)\Omega \tag{H.4}$$

The complex propagation is

$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{(2 + j100)(j0.01)} \\ &= j\sqrt{1 - j0.02} \\ &\approx j + 0.01 \end{aligned}$$

Hence the attenuation constant is 0.01 neper per metre and the phase constant is 1 radian per metre.

4. A *distortionless line* is defined as one in which α is independent of frequency and β is proportional to frequency. Show that this condition occurs when $R/L = G/C$.

Answer

From the basic propagation equations we have $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$. Under the given conditions this becomes

$$\begin{aligned} \alpha + j\beta &= j\omega\sqrt{LC\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)} \\ &= j\omega\sqrt{LC\left(1 + \frac{R}{j\omega L}\right)^2} \\ &= \left(j\omega + \frac{R}{L}\right)\sqrt{LC} \end{aligned}$$

Hence $\beta = \omega\sqrt{LC}$ and $\alpha = R\sqrt{\frac{C}{L}}$, i.e. α is independent of frequency and β is proportional to frequency. Thus all frequency components of a complicated waveform will travel with the same velocity, and all will suffer the same attenuation with distance. These features ensure that the wave propagates without distortion.

5. Calculate the input impedance of 1000 m of the transmission line described in Problem 3. Explain why this does not depend on the load impedance Z_L.

Answer

We observe that for the line in Problem 3 above, the product of the attenuation constant α and twice the line length is 20. Thus any forward wave launched at the sending end will, even if fully reflected at the load end, suffer attenuation by a factor of e^{20}, ie by about 174 dB, before it gets back to the input. Hence at the input, only V_f and I_f are present to significant amplitude. Hence the input impedance is

$$Z_{in} = \frac{\mathrm{V}_{in}}{\mathrm{I}_{in}} = \frac{\mathrm{V}_f}{\mathrm{I}_f} = Z_0 \tag{H.5}$$

6. An impedance of $(100 + j100)\ \Omega$ is placed as a load on a lossless transmission line of characteristic impedance 50 Ω. Find the reflection coefficient in magnitude and phase at the load end. What is its magnitude and phase at the input end if the line is $3\lambda/8$ long at the operating frequency? How does this impedance change as the frequency varies?

Answer

The voltage reflection factor is given by

$$\begin{aligned} \Gamma_v(L) &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= \frac{50 + j100}{150 + j100} \end{aligned}$$

$$\begin{aligned} &= \frac{112\angle 63.4^\circ}{180\angle 33.7^\circ} \\ &= 0.620\angle 29.7^\circ \end{aligned}$$

The voltage reflection factor at the input is the same in magnitude as at the outupt but is retarded in phase by 720° for each wave length of line. For a $3\lambda/8$ line, the phase retardation is 270°. Thus

$$\Gamma_v(S) = 0.620\angle 119.7^\circ$$

As the frequency is varied, we will assume in the absence of other information that Z_L will not change (a somewhat dubious assumption). Then $\Gamma_v(L)$ will not change, and $\Gamma_v(S)$ will change in phase, (but not in magnitude), becoming further retarded in phase as the line length increases. Whatever happens to $\Gamma_v(L)$, we can be sure that when the frequency is raised so that the line becomes a half wave in length, the input reflection factor becomes equal to the load reflection factor.

Some sensible discussion along these lines, but not necessarily incorporating all the points above, should attract full marks.

H.3 Homework 3

1. Calculate the attenuation constant α in nepers per metre, and the transmission loss in dB per metre, for an air-filled co-axial line of inner and outer conductor radii a and b, respectively, in terms of conductor material conductivity σ and the proportions of the line.

2. How does this result scale with frequency?

3. What are the losses for an air-filled copper co-axial line of inner conductor radius 2 mm and outer conductor radius 7 mm at frequencies of:

   (a) 100 kHz; and

   (b) 10 Mhz?

4. What length of this cable would you use as a high power dummy load to achieve, at a frequency of 1.0 GHz, and for any value of load impedance, an input VSWR of less than or equal to 1.5?

Answer

1. For the co-axial line in which the phasor representing the circumferential component of magnetic field at the inner conductor is $\mathrm{H_0}$, we have derived the results

$$W_L = \pi R_s |\,\mathrm{H_0}|^2 a(a+b)/b \tag{H.6}$$

$$W_T = \pi\eta|\ \mathrm{H}_0|^2 a^2 \log_e(b/a) \tag{H.7}$$

where W_L is the power lost in the walls per unit length of the line, W_T is the power transmitted along the line, and R_s is the surface resistivity due to skin effect. If σ is the material conductivity and δ is the skin depth, the R_s is given by

$$R_s = \frac{1}{\delta\sigma} = \sqrt{\frac{\omega\mu_0}{2\sigma}} \tag{H.8}$$

From conservation of energy applied to the length δz of coaxial line shown in Figure H.5 we conclude that

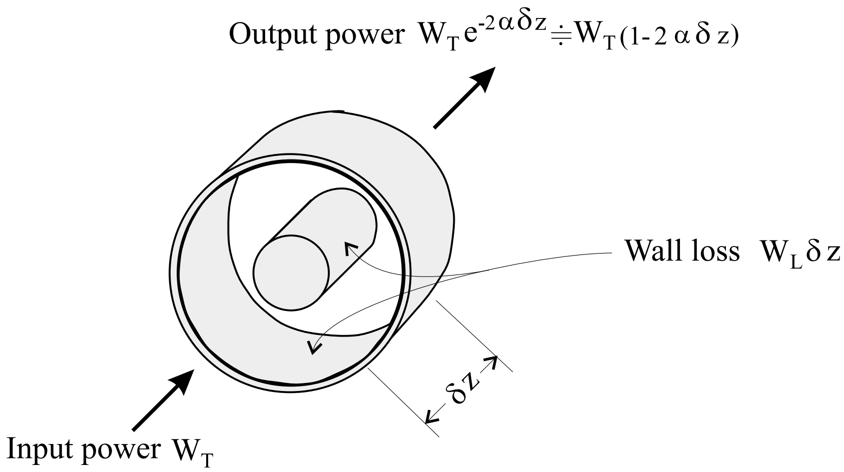


Figure H.5: Coaxial line section with wall loss.

$$\alpha = \frac{W_L}{2W_T} \tag{H.9}$$

Hence the desired expression for α is

$$\begin{aligned}\alpha &= \sqrt{\frac{\omega\mu_0}{2\sigma}}\frac{1}{\eta}\frac{a+b}{2ab\log_e(b/a)} \\ &= \sqrt{\frac{\omega\epsilon}{2\sigma}}\frac{a+b}{2ab\log_e(b/a)}\end{aligned}$$

This is the attenuation constant in nepers per metre. Now for one neper attenuation, we have an amplitude reduction factor of e, and a power reduction factor of e^2, which comes in more usual terms to 8.686 dB. Hence the loss of the line in dB per metre is

$$8.686\sqrt{\frac{\omega\epsilon}{2\sigma}}\frac{a+b}{2ab\log_e(b/a)} \quad \text{dB/m} \tag{H.10}$$

2. Clearly the loss scales as $\sqrt{\omega}$.

3. (a) For copper at 100 kHz and the indicated proportions of the line, we have

$$\begin{aligned}
\omega &= 2\pi \times 10^5 \ \text{rad/s} \\
\epsilon &= 8.854 \times 10^{-12} \ \text{F/m} \\
\sigma &= 5.8 \times 10^7 \ \text{S/m} \\
a &= 2 \ \text{mm} \\
b &= 7 \ \text{mm}
\end{aligned}$$

Hence $\alpha = 5.62 \times 10^{-5}$ neper/m i.e. 4.88×10^{-4} dB/m

(b) At 10 Mhz, we will have an attenuation ten times this amount, ie 4.88×10^{-3} dB/m.

4. To achieve a VSWR of ≤ 1.5 we must have a reflection coefficient $|\Gamma_v| \leq 0.2$. The power reflection coefficient is thus ≤ 0.04, i.e. the reflection loss for power sent down the line is 14 dB. The worst case corresponds to a lossless termination, in which case we need at least 7 dB of attenuation for each of the two passages (forward and reverse) of the signal along the line. Since at 1 Ghz, the loss is expected to be 0.0488 dB/m, we need $7/0.0488 = 143$ metres of cable.

H.4 Homework 4

Distributed:

Hand in:

1. A rectangular waveguide of dimensions 22.86 mm x 10.16 mm is operating in the dominant TE_{10} mode at a frequency of 10 GHz. Determine for this mode:

   (a) the cut-off frequency f_c,

   (b) the phase constant β,

   (c) the wavelength λ_g,

   (d) the phase velocity v_p,

   (e) the group velocity v_g,

   (f) the wave impedance Z_{TE}.

2. If the peak amplitude of the electric field in the waveguide above is 1.0 kV/m, calculate the power carried by the guide.

3. For the waveguide described above, calculate the attenuation constant in dB/m of the dominant mode at the frequency of 5 GHz, at which frequency that mode is below cut-off.

Answer

1. (a) The cut-off angular frequency ω_c of a TE_{1m} mode in rectangular waveguide is given by

$$\left(\frac{\omega_c}{c}\right)^2 = \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \tag{H.11}$$

.

For the TE_{10} mode with $l = 1$, $m = 0$, and $a = 22.86$ mm, we have $\omega_c = 4.1228 \times 10^{10}$ rad/s, from which we have $f_c = 6.562$ GHz.

(b) The propagation constant γ for a TE mode of a waveguide satisfies the relation

$$\gamma^2 = \left(\frac{\omega_c}{c}\right)^2 - \left(\frac{\omega}{c}\right)^2 \tag{H.12}$$

.

For $\omega > \omega_c$ we have a phase constant

$$\beta = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_c}{c}\right)^2}. \tag{H.13}$$

With $\omega = 2\pi \times 10$ GHz and ω_c as above we find $\beta = 158.05\ \text{m}^{-1}$.

(c) The wavelength $\lambda_g = 2\pi/\beta$ is then found to be 39.754 mm.

The group and phase velocities for a wave with angular frequency ω travelling in hollow waveguide in a mode for which the cut-off frequency is ω_c are given by

$$\begin{aligned} v_p &= \frac{\omega}{\beta} = \frac{c}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} \\ v_g &= \frac{\partial\omega}{\partial\beta} = c\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \end{aligned} \tag{H.14}$$

.

With the above values of ω and ω_c we find

(d) $v_p = 397.54$ m/s, and

(e) $v_g = 226.39$ m/s.

(f) The transverse field wave impedances Z_{TE} and Z_{TM} for rectangular waveguide are given by

$$Z_{TE} = (\beta_0/\beta)\eta \text{ and} \tag{H.15}$$

$$Z_{TM} = (\beta/\beta_0)\eta \tag{H.16}$$

.

With the above value of β, and $\beta_0 = 2\pi \times 10^{10}/(3 \times 10^8)$, and $\eta = 120\pi$, we find

$$Z_{TE} = 499.57\ \Omega. \tag{H.17}$$

2. For the calculation of the power flow, we note first that the waveguide fields for the dominant mode take the form

$$\begin{aligned}
\mathrm{E}_x &= 0 && \text{(H.18)}\\
\mathrm{E}_y &= -j\eta(\beta_0/\beta_c)\ \mathrm{H}\sin(\pi x/a)e^{-j\beta z} && \text{(H.19)}\\
\mathrm{E}_z &= 0 && \text{(H.20)}\\
\mathrm{H}_x &= j(\beta/\beta_c)\ \mathrm{H}\sin(\pi x/a)e^{-j\beta z} && \text{(H.21)}\\
\mathrm{H}_y &= 0 && \text{(H.22)}\\
\mathrm{H}_z &= \ \mathrm{H}\cos(\pi x/a)e^{-j\beta z} && \text{(H.23)}
\end{aligned}$$

The calculation of power flow down the waveguide requires the integration of the power density per unit area

$$S_z = -\frac{1}{2}E_y H_x^* = \frac{|E_y|^2}{2Z_{TE}} \tag{H.24}$$

over the waveguide cross section of width a and height b as illustrated in Figure H.6. This operation may be regarded as multiplying the power flow per unit area, averaged over the waveguide cross section, by the area ab of that cross section. The factor of 2 in the denominator of the above equation occurs because we are as usual using phasors which represent peak values.

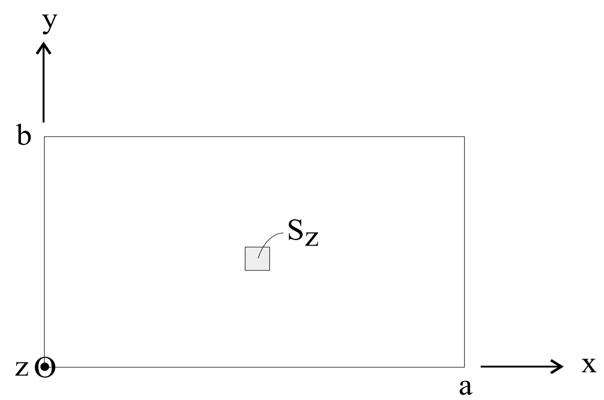


Figure H.6: Illustration of power flow in rectangular wavegiude.

From the expression for the waveguide fields given above we see that (i) the maximum power density per unit area occurs at the region half way between the side walls, (ii) that there is no variation of the power density with position along the

y axis, and (iii) that the variation with respect to the x axis has the form of the square of a sine wave.

From the well known result that the average value of the square of a sine wave is half the peak value, (this is the basis of the r.m.s. value of a sinusoid being $1/sqrt2$ of the peak value), we conclude that the power flow down the waveguide is half the maximum value of the power flow per unit area times the area of the waveguide.

If E_{max} is the peak value of the electric field at the position of maximum field, the power flow per unit area at that position is

$$S_z = \frac{E_{max}^2}{2Z_{TE}} \tag{H.25}$$

.

The average power flow per unit area, averaged over the waveguide cross section, is in the light of the preceding discussion

$$S_{z\ avg} = \frac{E_{max}^2}{4Z_{TE}} \tag{H.26}$$

and hence the total power flow over the area ab is

$$P = \frac{E_{max}^2 ab}{4Z_{TE}} \tag{H.27}$$

.

Substituton of the values $E_{max} = 1$ kV/m, $a = 22.86$ mm, $b = 10.15$ mm, and the value of $Z_{TE} = 499.57\,\Omega$ obtained above gives

$$P = 116.2 \text{ mW} \tag{H.28}$$

.

3. The propagation constant γ for a TE mode of a waveguide satisfies the relation

$$\gamma^2 = \left(\frac{\omega_c}{c}\right)^2 - \left(\frac{\omega}{c}\right)^2 \tag{H.29}$$

.

When the mode is the dominant mode and hence $\omega_c = 2\pi \times 6.562$ GHz as above, but ω is only $2\pi \times 5$ GHz, we find $\gamma = \alpha$ is real and equal to 89 neper per metre.

Converting this result to dB per metre involves multiplying by 2 and then by $10 \log_{10} e$, i.e. multiplying by 8.68, to give an attenuation of 772.5 dB/m.

This is a very large attenuation with distance indeed, sufficient, for example, to attenuate in a length of only 200 mm of waveguide, even watt level signals to far below thermal noise for even a GHz of bandwidth.

Appendix I

TUTORIAL ANSWERS

In this Appendix, the original questions are reproduced along with the answers. The section numbering used in this Apendix follows that used in Appendix G wherein the questions originally appeared.

I.1 Tutorial 1

1. For the coaxial transmission line shown in Figure I.1, in which the centre conductor has dc current i, and steady charge q per unit length, use the integral forms of Gauss' and Ampere's laws to calculate the values of the electric field and the magnetic field at a radius r. Assume that the centre conductor is supported by a non-magnetic dielectric of dielectric permittivity ϵ.

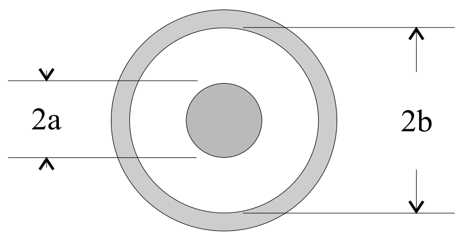


Figure I.1: Co-axial line configuration.

Answer

In respect of this question, we could assume a short length δz of line, a line charge density q per unit length on the inner conductor, and a current i on that conductor. Then by employing Gauss' law of electrostatics, an expression for the radial component of electric flux density, and then the electric field, at a distance r from the centre can be derived. By employing Ampere's law of magnetostatics, an expression for the circumferential component of the magnetic field, and then the magnetic flux density, at a distance r from the centre can be derived.

Integration of the electric field expression, with due care to signs and the order of limits of the integral, should then give the potential of the inner conductor relative to the outer conductor, and should lead, after this result is divided into the charge per unit length, to an expression for the capacitance per unit length.

Calculation of the flux linked by a surface of length δz and width from the inner to outer conductor, and dividing by the current i, should give the inductance of the line. The results should be

$$C = \frac{2\pi\varepsilon}{\log_e\left(\frac{b}{a}\right)}$$
$$L = \frac{\mu_0}{2\pi}\log_e\left(\frac{b}{a}\right)$$

2. A lossless transmission line, initially charged to a dc voltage V, is shorted at its input at $t = 0$. Sketch the voltage $v(t)$ at the open-circuit output end.

Answer

Let T be the time for a wave to propagate one way along the length L of the line. Before the switch is closed the forward and backward waves are $V_f(z,t) = \frac{1}{2}$ V, $V_r(z,t) = \frac{1}{2}$ V, so that everywhere on the line $v(z,t) =$ V.

The discussion is slightly simplified by putting the switch on right, as:

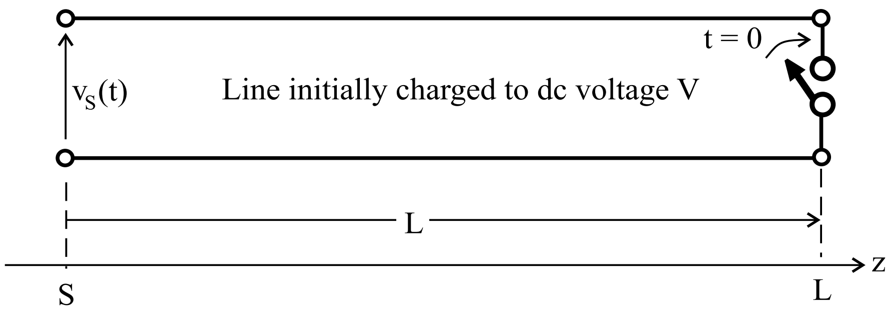


Figure I.2: Charged line with shorting switch.

When the switch is closed $V_f(S,t)$ will initially remain the same, and so will $V_r(S,t)$ at least for $0 \le t \le T$. At the load end $V_f(L,t)$ remains the same initially, and $V_r(L,t)$ becomes $-\frac{1}{2}$V to produce zero total $v(L,t)$ at that end. After time T, the reverse wave of $-\frac{1}{2}$V arrives at $z = 0$. At this time $V_f(S,t)$ changes to $-\frac{1}{2}$V as well, because $V_f(S,t)$ may be considered as arising from $V_r(S,t)$ by reflection from the open circuit input end, and the reflection factor for an open circuit is $\Gamma_v = 1$. Continuing in this way we see the waveforms are as given in Figure I.3

3. A pulse generator consists of 10 metres of 50 Ω transmission line in which the velocity is 2×10^8 m/s, charged to 6 kV dc, and connected to a load impedance of 25 Ω by a relay which is closed at time $t = 0$. Sketch to definite scales the output pulse obtained. Calculate the peak output power and the total output energy in the pulse.

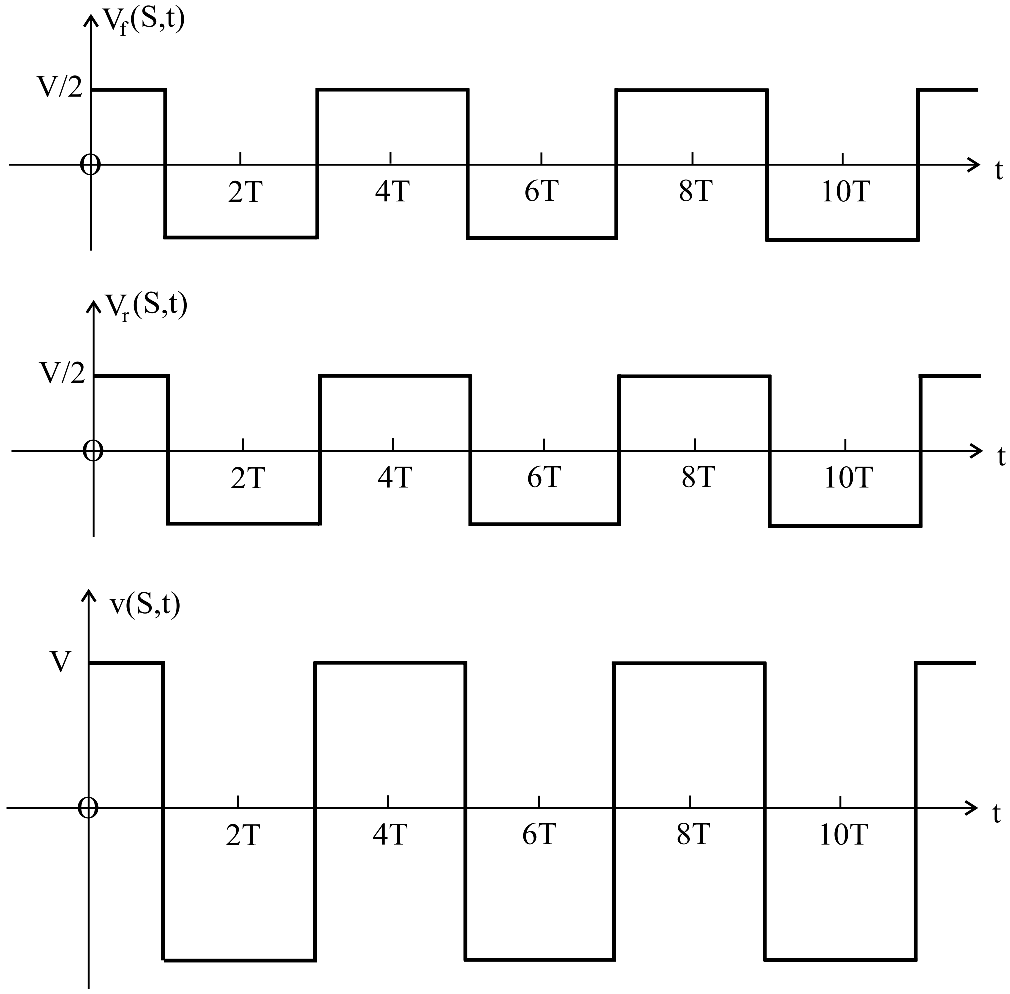
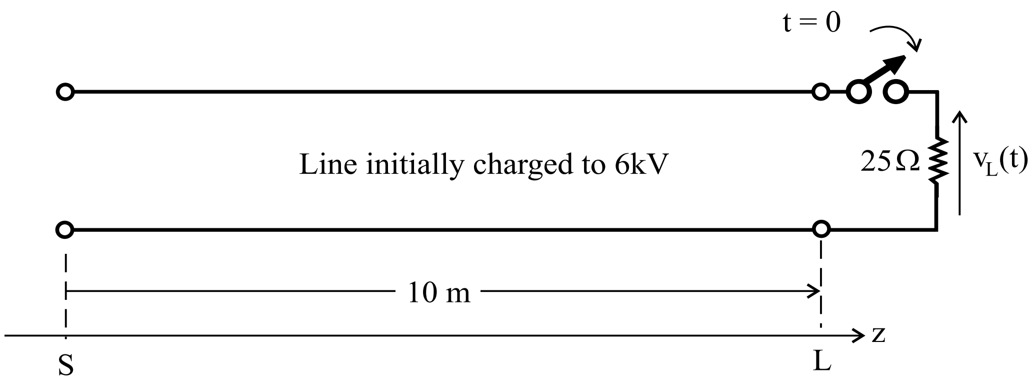


Figure I.3: Waveforms for charged line with shorting switch.

Answer



The propagation time T is 50 ns. The voltage reflection factors are at the sending end $\Gamma_v(S) = +1$ and at the load end $\Gamma_v(L) = -\frac{1}{3}$. Relations between V_f and V_r established at the ends of the line for $t > 0$ are at the sending end $V_f(S,t) = V_r(S,t)$ and at the load end $V_r(L,t) = -\frac{1}{3}V_f(L,t)$. Analysis of sucessive reflections gives the waveforms shown in Figure 3.

The peak output power, which occurs in the time interval $0 \leq t \leq 2T$, is given by

$$\begin{aligned} P_0 &= \frac{2000^2}{25} \\ &= 160 \ \ \text{kW} \end{aligned}$$

The calculation of total output energy may be done in two ways. Firstly we may note that the output voltage in the nth time interval of length $2T$ is, using the notation V for the initial voltage of 6 kV on the line,

$$v_n = -V\left(\frac{-1}{3}\right)^n \tag{I.1}$$

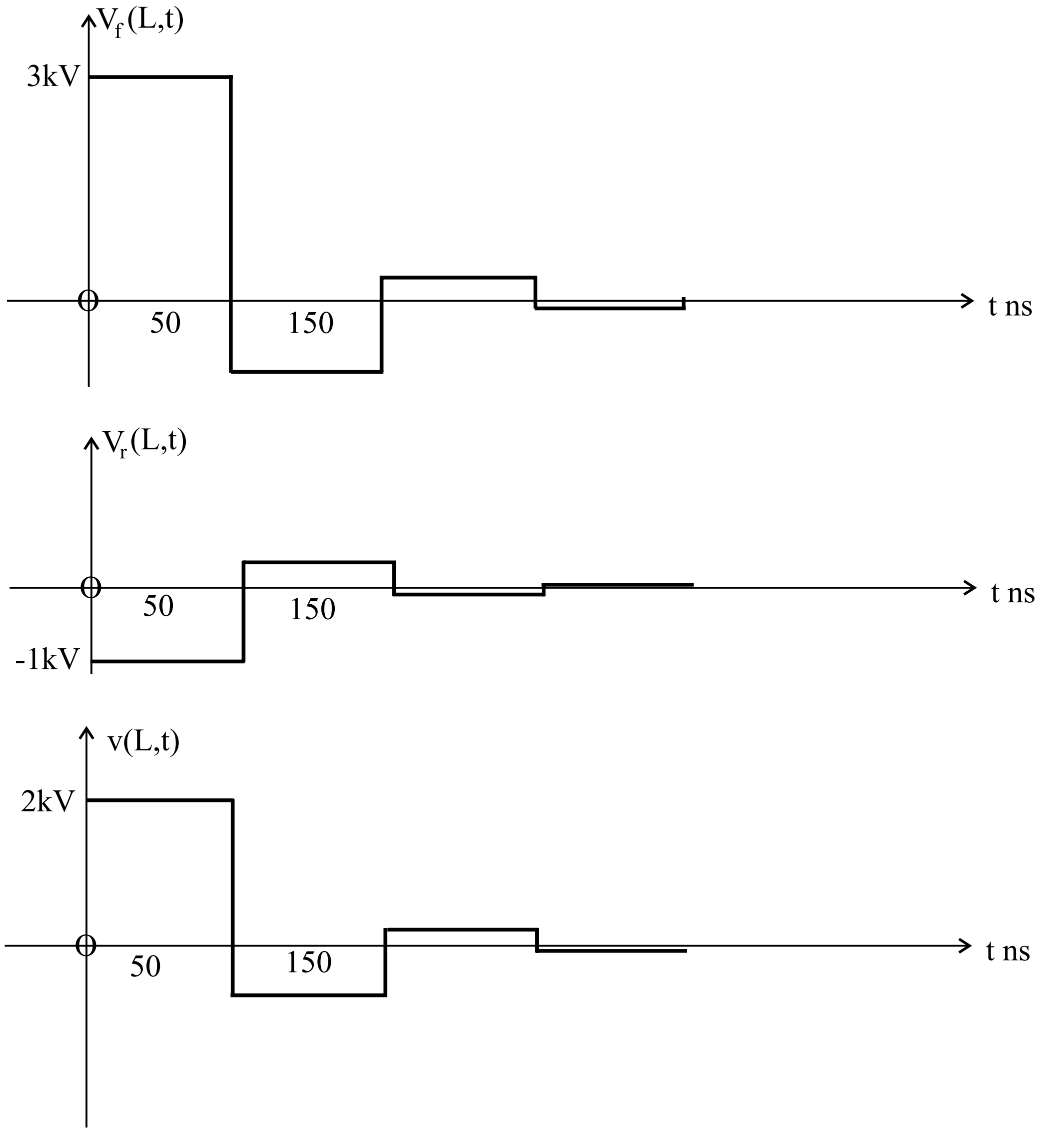
the output energy in that interval is given by

$$W_n = \frac{V^2}{R_L}\left(\frac{1}{9}\right)^n 2T \tag{I.2}$$

The sum to infinity of these energies is

$$\begin{aligned} W_T &= \frac{V^2}{R_L}\left(\frac{1}{8}\right)2T \\ &= \frac{V^2 T}{2Z_0} \end{aligned}$$

where we have used $R_L = Z_0/2$.


$V_f(L,t)$
3kV
0
50
150
t ns
$V_r(L,t)$
0
50
150
t ns
-1kV
v(L,t)
2kV
0
50
150
t ns

As an alternative, we may note that the total output energy is the energy stored initially in the line, and that if R_L were changed to Z_0 this energy would be simply output in the form of a single pulse of amplitude $V/2$ and duration $2T$ across a resistor Z_0. The total output energy in such a pulse is

$$
\begin{aligned}
W_T &= \left(\frac{V}{2}\right)^2 \frac{2T}{Z_0} \\
&= \frac{V^2 T}{2Z_0}
\end{aligned}
$$

as before. With the numerical values given we find that $W_T = 18$ mJ.

I.2 Tutorial 2

1. In the circuit in Figure I.4, the transmission line has air dielectric, and the resistors are lumped elements.

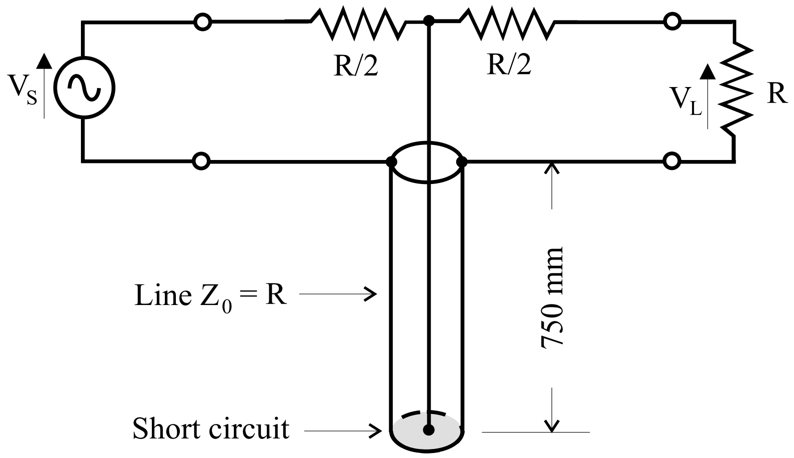


Figure I.4: Transmission line in a lumped parameter circuit.

(a) Sketch the frequency response of the circuit over the frequency range from 0 to 500 MHz.

(b) Sketch the frequency response when the short circuit on the transmission line is replaced by an open circuit.

Answer

(a) The shorted end transmission line will appear at its input end as

(a) a s/c at dc;

(b) an o/c at 100 MHz;

(c) a s/c at 200 MHz;
(d) an o/c at 300 MHz;
(e) a s/c at 400 Mhz; and
(f) an o/c at 500 MHz.
The approximate frequency response is shown in Figure I.5.

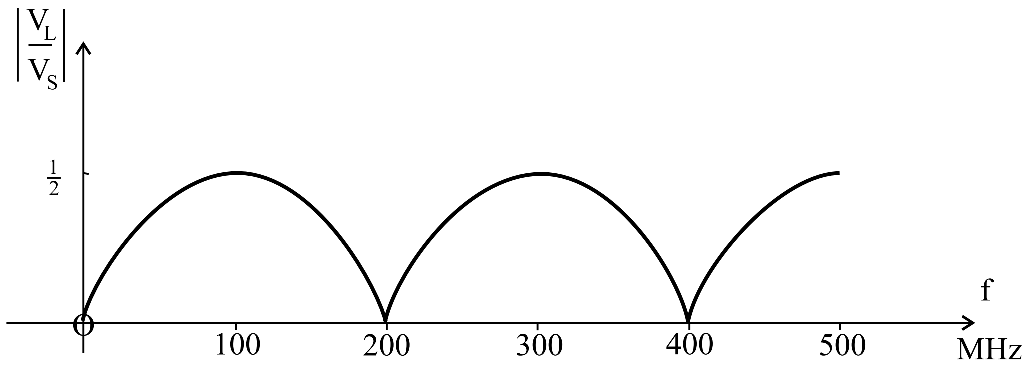


Figure I.5: Frequency response for circuit with shorted line.

(b) The open circuited end transmission line will appear at its input end as
(a) an o/c at dc;
(b) a s/c at 100 MHz;
(c) an o/c at 200 MHz;
(d) a s/c at 300 MHz;
(e) an o/c at 400 Mhz; and
(f) a s/c at 500 MHz.
The approximate frequency response is shown in Figure I.6.

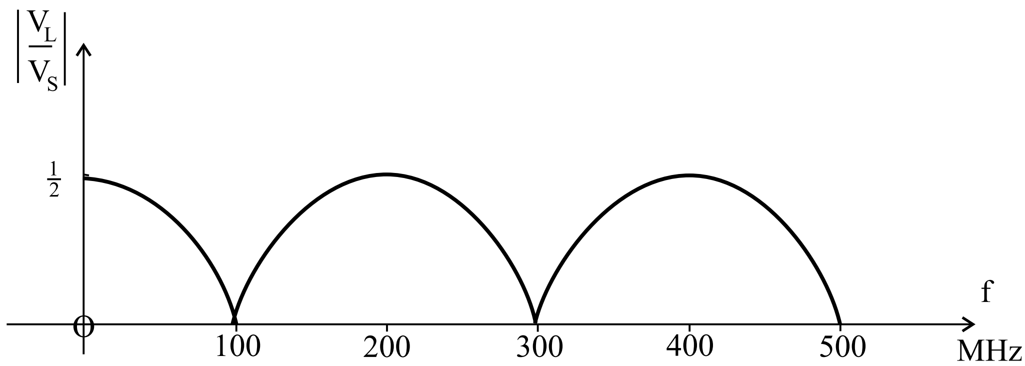


Figure I.6: Frequency response for circuit with open circuit line.

2. (a) At a frequency of 1000 MHz, to what capacitance is 10 cm of 50 Ω line, shorted at its end, equivalent? Assume that the dielectric is air.

(b) If the line has instead a dielectric with dielectric constant 2.25, what is the capacitance?

Answer

(a) For an air line, $c = 3\times10^8$m/s. Hence $\beta = \omega/c = (20\pi/3)\text{m}^{-1}$ at 1 GHz. Hence $\beta l = 2\pi/3$ for $l = 0.1$m. For a shorted line, $Z_I = jZ_0\tan(\beta l) = -86.6j\Omega$ when $\beta l = 2\pi/3$. The corresponding capacitance is given by

$$\begin{aligned} C &= \frac{1}{86.6\omega} \\ &= \frac{1}{86.6\times 2\pi\times 10^9} \\ &= 1.84 \ \text{pF} \end{aligned}$$

(b) If the dielectric constant is 2.25, the capacitance per unit length is increased by this factor. The following changes result in the line parameters:
(i) β is increased by a factor $\sqrt{2.25} = 1.5$; and
(ii) Z_0 is decreasd by a factor $\sqrt{2.25} = 1.5$.
With the new value of β and the old line length of 0.1m, the product βl now has become π, i.e. the line is now half a wave length long. The input impedance is thus equal to the load impedance, ie a short circuit, and the equivalent capacitance becomes infinite.

3. What is the input impedance at a single frequency for the transmission lines shown in Figure I.7, expressed as a large or small inductance or capacitance? In Figure I.7, δ is a length very much less than the wave length λ on the line at the frequency of interest.

Answer

For the shorted lines, we make use of the relation $Z_I = jZ_0\tan(\beta l)$, which tells us that a shorted line of length much less than a quarter wave length is inductive and a low impedance, and that as the length increases it remains inductive but of increasing impedance until the length reaches a quarter wavelength, after which it becomes capacitive but initially of a high impedance, the impedance decreasing but remaining capacitive until a half wave length is reached at which point the impedance has reduced to zero. For lines of length greater than a half wave lenth, the behaviour described above repeats, ie we can take any integral number of half wave lengths off the line length.

For the open circuit lines, we make use of the relation $Z_I = -jZ_0\cot(\beta l)$, which tells us that an open circuit line of length much less than a quarter wave length is capacitive and a high impedance, and that as the length increases it remains capacitive but of decreasing impedance until the length reaches a quarter wavelength, after which it becomes inductive but initially of a low impedance, the impedance increasing but remaining inductive until a half wave length is reached at which point the input has become an open circuit. For lines of length greater than a half wave

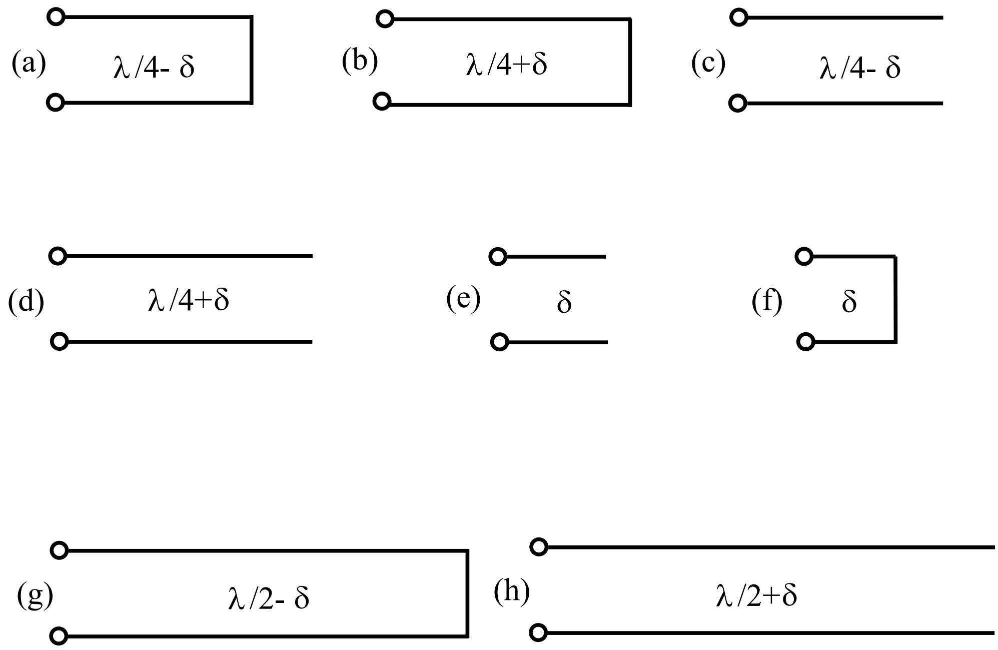


Figure I.7: Various open and short circuit transmisison lines.

lenth, the behaviour described above repeats, ie we can take any integral number of half wave lengths off the line length.

Using these insights, and noting that a small capacitance has a large impedance, we see that the illustrated lines behave as:

(a) a large inductance;

(b) a small capacitance;

(c) a large capacitance;

(d) a small inductance;

(e) a small capacitance;

(f) a small inductance;

(g) a large capacitance; and

(h) a small capacitance.

4. What current flows in the short circuit at the end of the lossless line shown in Figure I.8?

Answer

The line appears as an open circuit at the input. Hence a voltage of 2V appears at the input. Thus, at the input the forward and backward waves V_f and V_r are in phase and each equal to V. A the load end, V_f will be retarded in phase by $\pi/2$ ie will be $-j$V, and V_r will be advanced in phase by $\pi/2$ ie it will be jV. The current

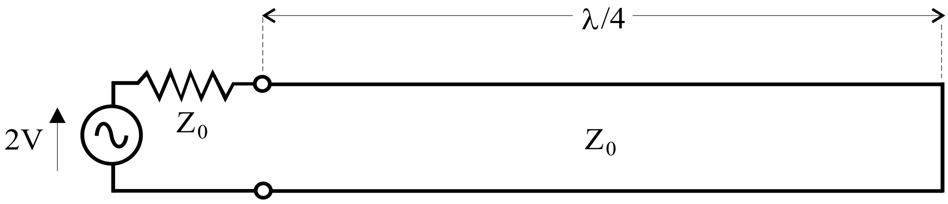


Figure I.8: Shorted lossless transmision line with matched source.

I will be $(\mathrm{V}_f - \mathrm{V}_r)/Z_0 = -2j\ \mathrm{V}/Z_0$. Note that this current will be obtained for any value of resistor in series with the generator, not merely the value shown.

5. If a coaxial cable with loss 0.2 dB per metre is used as in Figure I.9, what is the VSWR:

   (a) at the load end; and

   (b) at the source?

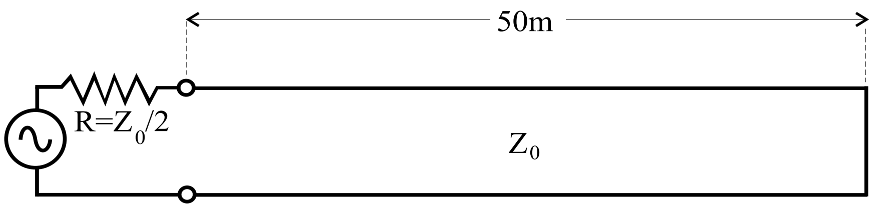


Figure I.9: Shorted lossy transmision line with unmatched source.

Answer

At the load end of the line we have $\Gamma_v = -1$ and $|\Gamma_v| = 1$. The VSWR is infinite. At the source end, we can assert that any forward wave sent down the line will arrive back attenuated by 20 dB. Thus at the source end $|\mathrm{V}_r| = 0.1|\mathrm{V}_f|$. The VSWR is then

$$\begin{aligned} S &= \frac{|\mathrm{V}_f| + |\mathrm{V}_r|}{|\mathrm{V}_f| - |\mathrm{V}_r|} \\ &= 1.22 \end{aligned}$$

I.3 Tutorial 3

1. A 75 Ω line is terminated in $(50 + j100)$ Ω.

   (a) What is the VSWR? What percentage of forward power is reflected from the load?

(b) What is the distance from the load back to the first voltage minimum, and to the first voltage maximum?

(c) If the line is to be matched by a single stub, what is its length and position? Both the line and the stub have characteristic impedance $Z_0 = 75\ \Omega$.

Answer

(a)

$$\begin{aligned} z_L &= 0.667 + j1.333 \\ |\Gamma_v| &= 0.643 \\ \text{VSWR} &= 4.6 \\ \%\text{ of reflected power} &= 41\% \end{aligned}$$

(b)

$$\begin{aligned} \text{Distance to } V_{min} &= 0.34\,\lambda \ \text{ from load} \\ \text{Distance to } V_{max} &= 0.09\,\lambda \ \text{ from load} \end{aligned}$$

(c)

$$\begin{aligned} \text{Stub length} &= 0.083\,\lambda \\ \text{Stub position} &= 0.274\,\lambda \end{aligned}$$

2. In the matching system shown in Figure I.10, l_1 and l_2 are line lengths and s_1 and s_2 are the lengths of the stubs.

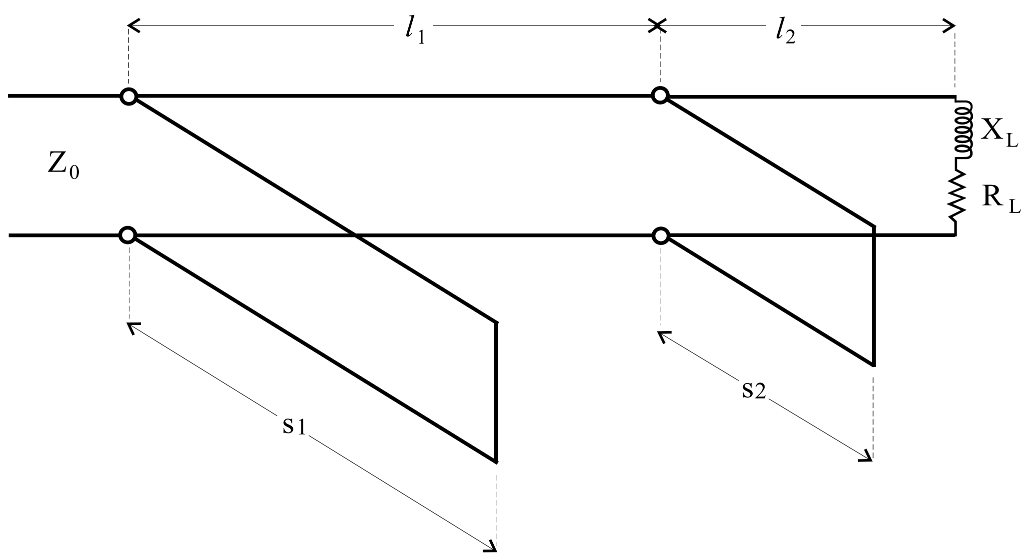


Figure I.10: A double stub matching system.

(a) If $l_2 = 0$ and $l_1 = 3\lambda/8$, what values of R_L cannot be matched?

(b) If $l_2 = 0$ and $l_1 = \lambda/4$, what values of R_L cannot be matched?

(c) If $l_2 = \lambda/10$ and $l_1 = 3\lambda/8$, and $R_L = Z_0/2$ and $X_L = Z_0$, what stub lengths s_1 and s_2 are required for a match?

(d) What is then the VSWR in line l_2, and what is it in line l_1?

Answer

(a) The values of R_L which cannot be matched with this arrangement depend on X_L, in that points of the Smith chart anywhere inside the circle whose diameter is the pair of points given (perhaps inappropriately, as the Smith chart is really a plot of voltage reflection factor or current reflection factor) in impedance terms by $z = 0 + j0$ and $z = 0.5 + j0$, cannot be matched.

(b) The values of R_L which cannot be matched with this arrangement again depend on X_L, in that points of the Smith chart anywhere inside the circle whose diameter is the pair of points given (perhaps inappropriately, as the Smith chart is really a plot of voltage reflection factor or current reflection factor) in impedance terms by $z = 0 + j0$ and $z = 1.0 + j0$, cannot be matched.

(c) We use an admittance chart because the stubs are in shunt. First we calculate $z_L = 0.5 + j1.0$ and then find that $y_L = 0.4 - j0.8$. When transformed along the line by $\lambda/10$ this becomes $y'_L = 0.24 - j0.1$. We then find $y_X = y'_L + y_{S2} = 0.24 - j0.35$; hence y_{S2} has the value $-j0.25$ so stub S2 then has length 0.211 λ. The point $y'_L + y_{S2}$ when transformed $3\lambda/8$ toward the generator becomes $y_Y = 1.0 - j1.72$. Thus y_{S1} needed for a match is $+j1.72$. The stub length which provides a normalised susceptance of 1.72 has length 0.417 λ.

(d) The VSWR in the line of length l_2 can be found from y_L or from y'_L to be 4.2. The VSWR in the line of length l_1 can be found from $y_X = y'_L + y_{S2}$ or alternatively from y_Y to be 4.6.

Appendix K

EXERCISES

K.1 Exercises on Notation

1. Construct phasors to represent:

   (a) $v(t) = 10\cos(\omega t + \pi/2)$ V; and

   (b) $i(t) = 5\sin\omega t$ A.

2. Find the real voltage as a function of position and time represented by the phasor $v(z) = \mathrm{V}_f e^{-j\beta z}$ where $\mathrm{V}_f = (3 + 3j)$ V.

K.2 Transmission Line Fields

For the coaxial transmission line shown in Figure K.1, in which the centre conductor has dc current i, and steady charge q per unit length, use the integral forms of Gauss' and Ampere's laws to calculate the values of the electric field and the magnetic field at a radius r. Assume that the centre conductor is supported by a non-magnetic dielectric of dielectric permittivity ϵ.

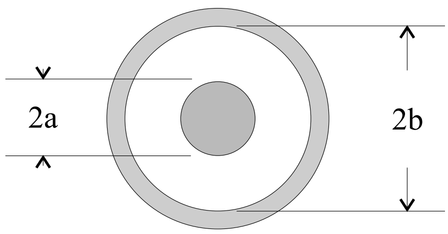


Figure K.1: Co-axial line configuration.

K.3 Transients on Transmission lines

1. A lossless transmission line, initially charged to a dc voltage V, is shorted at its input at $t = 0$. Sketch the voltage $v(t)$ at the open-circuit output end.

2. A pulse generator consists of 10 metres of 50 Ω transmission line in which the velocity is 2×10^8 m/s, charged to 6 kV dc, and connected to a load impedance of 25 Ω by a relay which is closed at time $t = 0$. Sketch to definite scales the output pulse obtained. Calculate the peak output power and the total output energy in the pulse.

3. A d.c. voltage source of internal voltage 1200 volts and internal resistance 10 Ω is connected at $t = 0$ to 10 metres of lossless co-axial transmission line. Measurements at low frequencies show that the length of line has a total capacitance of 1.0 nF and a total inductance of 2.5 μH. The line is teminated in a load resistance of 30 Ω.

   (a) Determine the characteristic impedance and the wave velocity for the coaxial line.

   (b) Draw a schematic circuit diagram and below it a lattice diagram of the forward and reverse waves of voltage.

   (c) Determine the transit time for a wave front to travel from the source to load and the reflection coefficients for each end of the line, and show them on the lattice diagram.

   (d) Show details of the voltage distribution as a function of position along the line at 1.5 and 2.5 times the one-way transit time.

   (e) Show details of the voltage waveform as a funtion of time at the load terminals.

   (f) What is the steady state voltage distribution for this line with the source and load specified?

K.4 Transmisson Lines in the Frequency Domain

1. Calculate the inductance per unit length of an air-cored co-axial line of radii a (inner conductor) and b (outer conductor).

2. What is the input impedance of each of the lossless lines shown in Figure K.2?

3. A transmission line has the following distributed parameters per unit length:

   $L = 0.5$ μH/m, $R = 2.0$ Ω/m, $C = 50$ pF/m, and $G = 0$ S/m.

   Calculate at a frequency of 31.8 MHz the characteristic impedance, and the attenuation and phase constants.

4. A *distortionless line* is defined as one in which α is independent of frequency and β is proportional to frequency. Show that this condition occurs when $R/L = G/C$.

5. Calculate the input impedance of 1000 m of the transmission line described in Problem 3. Explain why this does not depend on the load impedance Z_L.

6. An impedance of $(100 + j100)$ Ω is placed as a load on a lossless transmission line of characteristic impedance 50 Ω. Find the reflection coefficient in magnitude and phase at the load end. What is its magnitude and phase at the input end if the line is $3\lambda/8$ long at the operating frequency? What happens as the frequency varies?

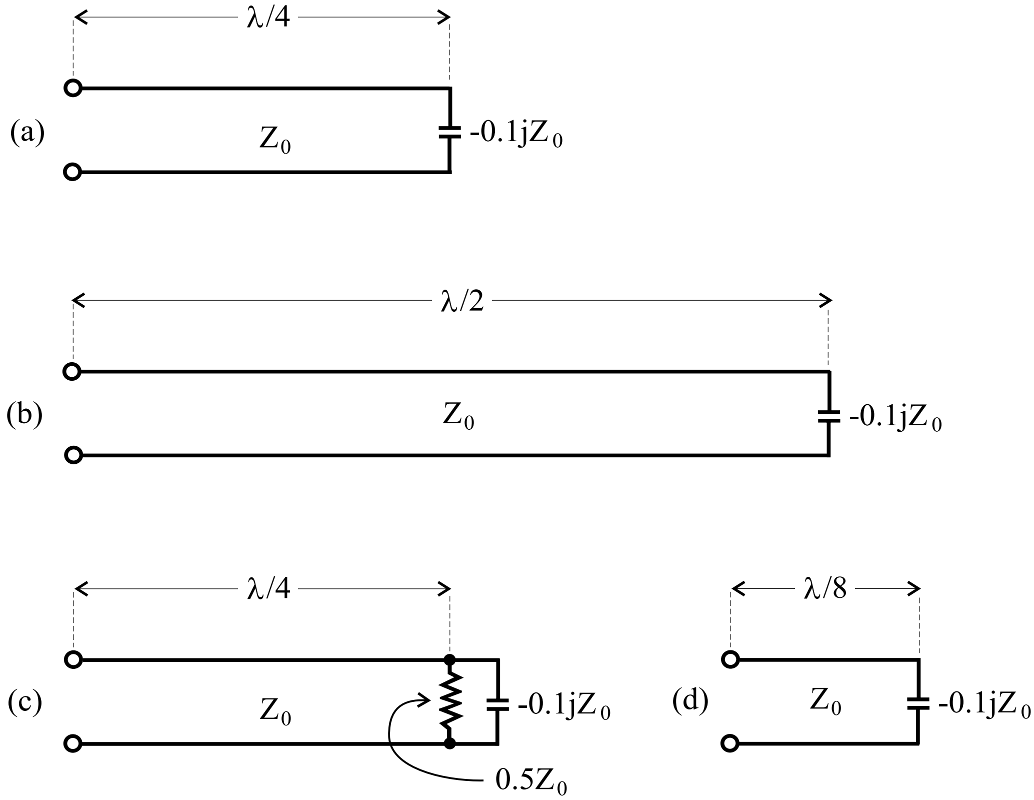


Figure K.2: Lossless transmission lines.

7. In the circuit in Figure K.3, the transmission line has air dielectric, and the resistors are lumped elements.

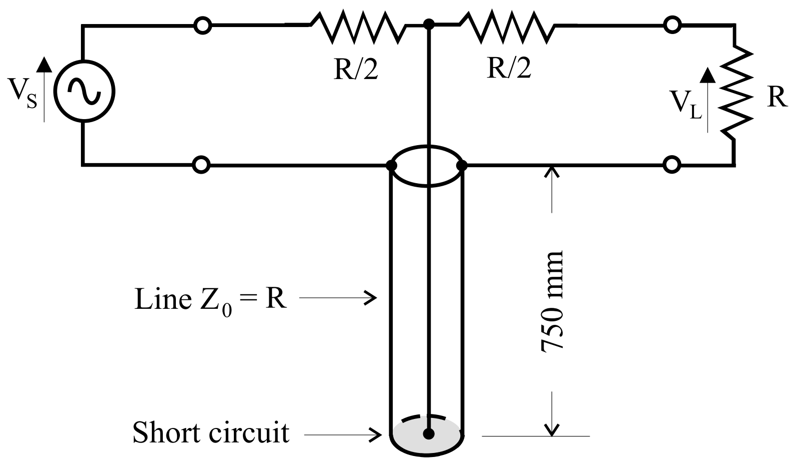


Figure K.3: Transmission line in a lumped parameter circuit.

   (a) Sketch the frequency response of the circuit over the frequency range from 0 to 500 MHz.
   (b) Sketch the frequency response when the short circuit on the transmission line is replaced by an open circuit.

8. (a) At a frequency of 1000 MHz, to what capacitance is 10 cm of 50 Ω line, shorted at its end, equivalent? Assume that the dielectric is air.
   (b) If the line has instead a dielectric with dielectric constant 2.25, what is the capacitance?

K.5 Miscellaneous Transmision Line Problems

1. What is the input impedance at a single frequency for the transmission lines shown in Figure K.4, expressed as a large or small inductance or capacitance? In Figure K.4, δ is a length very much less than the wave length λ on the line at the frequency of interest.

2. A transmission line of $Z_0 = 50\ \Omega$ is terminated in $Z_L = (100 + j100)\ \Omega$. Find the VSWR on the line, and the location of the first voltage minimum.

3. If a twin wire balanced telephone line is required to have a characteristic impedance of 600 Ω, calculate the required separation between conductors if enamel insulated copper wire 1 mm diameter is to be used.

4. A length of 80.62 cm of 50 Ω coaxial cable, in which the velocity is 2×10^8 m/s, is being used as a measurement cable at a frequency of 50 MHz on an oscilloscope which has an input capacitance of 20 pF, and negligible input conductance. Calculate the input impedance of the free end of the cable.

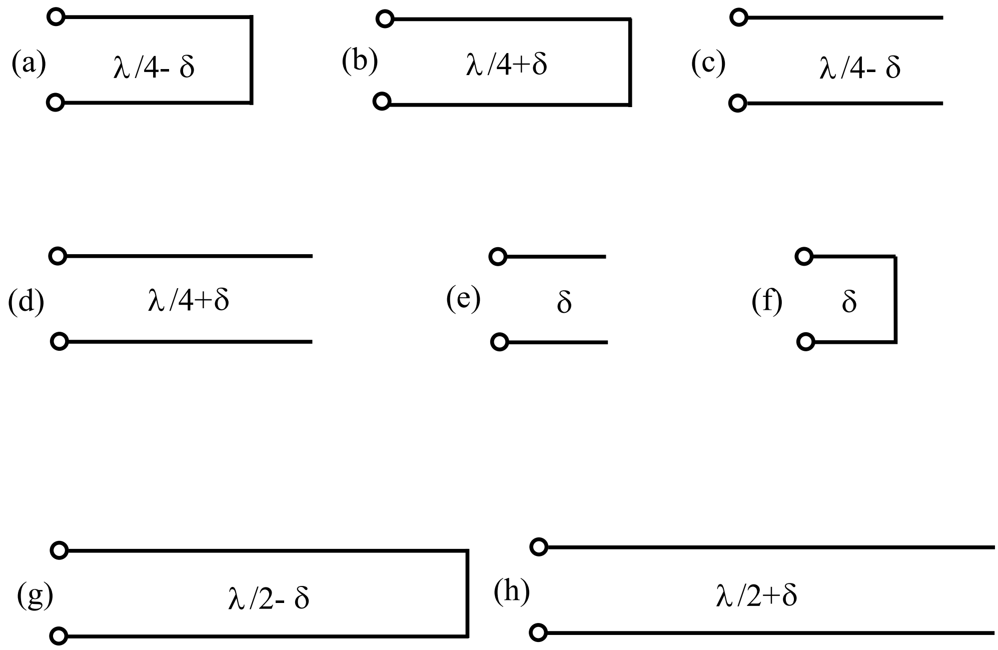


Figure K.4: Various open and short circuit transmisison lines.

5. A 50 Ω line is terminated in an impedance of $(75 - j69)$ Ω. The line is 3.5 m long and is excited by a source of energy at 50 MHz. The velocity of propagation is 3×10^8 m/s. Use the Smith chart to find the input impedance, the magnitude and phase of the input reflection coefficient, the standing wave ratio on the line, and the position of a voltage minimum.

6. The standing wave ratio on an ideal 70 Ω line is measured as 3.2, and a voltage minimum is observed 0.23 λ in front of the load. Use the Smith chart to find the load impedance.

K.6 Transmission Line Matching

1. What current flows in the short circuit at the end of the lossless line shown in Figure K.5?

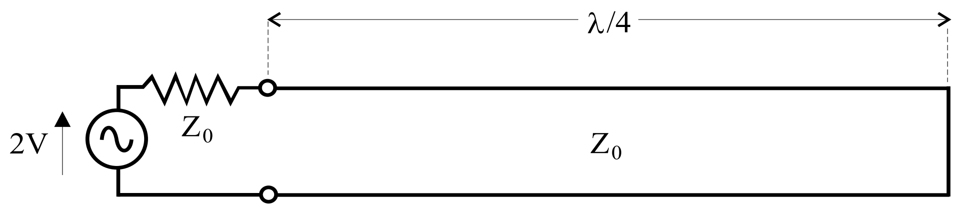


Figure K.5: Shorted lossless transmision line with matched source.

2. If a coaxial cable with loss 0.2 dB per metre is used as in Figure K.6, what is the VSWR

   (a) at the end; and

   (b) at the source?

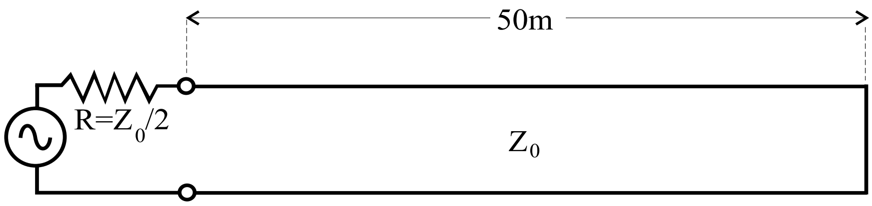


Figure K.6: Shorted lossy transmision line with unmatched source.

3. Using single stub matching, how would you match a transistor, the output impedance of which is $(20 - \jmath 30)\ \Omega$, to an antenna of impedance $(50 + \jmath 20)\ \Omega$ via a long length of low loss 50 Ω cable, at a frequency of 300 MHz?

4. A 50 Ω transmission line is terminated with a load $Z_L = (20 + \jmath 30)\ \Omega$. A double stub tuner consisting of a pair of shorted 50 Ω transmission lines connected in shunt with the main line and spaced 0.25 λ apart is located with one stub 0.2 λ from the load. Find the lengths of the stubs to give unity VSWR at 0.55 λ from the load, i.e. at the input to the tuner.

5. A 75 Ω line is terminated in $(50 + \jmath 100)\ \Omega$.

   (a) What is the VSWR? What percentage of forward power is reflected from the load?

   (b) What is the distance from the load back to the first voltage minimum, and to the first voltage maximum?

   (c) If the line is to be matched by a single stub, what is its length and position? Both the line and the stub have characteristic impedance $Z_0 = 75\ \Omega$.

6. In the matching system shownin Figure K.7, l_1 and l_2 are line lengths and s_1 and s_2 are the lengths of the stubs.

   (a) If $l_2 = 0$ and $l_1 = 3\lambda/8$, what values of R_L cannot be matched?

   (b) If $l_2 = 0$ and $l_1 = \lambda/4$, what values of R_L cannot be matched?

   (c) If $l_2 = \lambda/10$ and $l_1 = 3\lambda/8$, and $R_L = Z_0/2$ and $X_L = Z_0$, what lengths l_1 and l_2 are required for a match?

   (d) What is then the VSWR in line l_2, and what is it in line l_1?

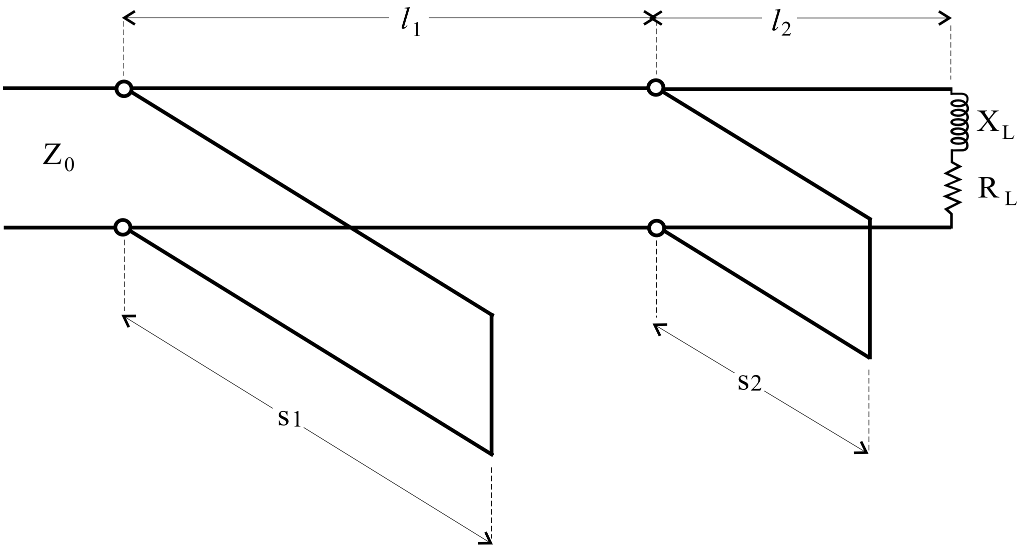


Figure K.7: A double stub matching system.

K.7 Plane Wave Problem

A linearly polarised uniform plane wave propagating at a frequency of 100 MHz in the z direction carries a power of 10 μWm^{-2}. Calculate the real and complex Poynting vectors, and write expressions as a function of position for the phasors representing the electric and magnetic fields. Hence derive expressions as a function of position and time for the real field vectors.

K.8 Transmission Line Interpretation of Wave Reflection

A linearly polarised unform plane wave is incident as shown in Figure K.8 from the left on the slab of lossless dielectric material of thickness L.

1. Calculate the characteristic impedance and propagation constant for free space and for the medium.

2. Draw a transmission line analogy for the reflection and transmission situation above. Assuming the frequency is 50 MHz and the length L is 0.5 m, use a Smith chart to determine the normalized impedance at the input to the slab, and hence the fraction of the input power reflected.

3. Determine also the fraction of the original incident power which is transmitted to free space on the other side of the slab.

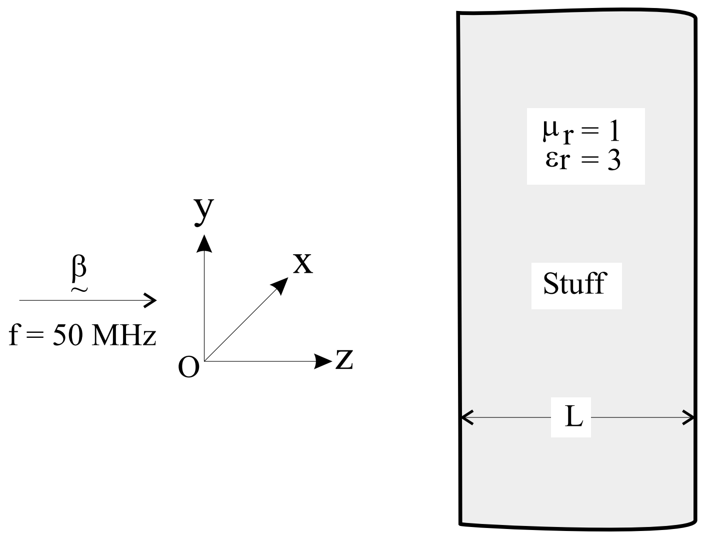


Figure K.8: Transmission through a lossless dielectric slab.

K.9 Rectangular Waveguide Problems

1. A standard X-band waveguide, normally used for frequencies in the band 8.2 to 12.4 GHz, has interior dimensions a = 22.86 mm and b = 10.16 mm. Calculate the cut-off frequency of the dominant, ie. the TE_{10} mode.

2. Derive an expresson for the cut-off frequencies of the TE_{lm} and TM_{lm} modes, of the form:

$$f_c = B\sqrt{Al^2 + m^2} \tag{K.1}$$

   where B is a coefficient which has the units of Hz, and A is a dimensionless coefficient.

   Hence prepare a *mode chart* showing the cut-off frequencies of the five lowest TE modes.

3. Establish the constraint on the height-to-width ratio of a rectangular waveguide which will maximise the ratio between the cut-off frequencies of the dominant mode and the next highest mode, the latter frequency being in the numerator.

4. Investigate the manner in which:

   (a) the attentuation; and

   (b) the power carrying capacity

   of a waveguide of given width is affected by changes in waveguide height, at a particular operating frequency. Hence derive the height to width ratio which will simultaneously satisfy the constraint established in part (3) above, and which will:

(c) minimise the attenuation; and

(d) maximise the power.

K.10 Conductor Classification

The conductivity of graphite is about 0.12 S/m. Taking its dielectric constant as 5, find an approximate frequency range over which it might be classed as a good conductor.

K.11 Plane Wave Reflection From a Perfect Conductor

A uniform plane wave, polarised as shown in Figure K.9, is incident at an angle θ on the plane boundary $z = 0$ of a perfect conductor. Derive an expression for the real surface current density K, as a function of position and time, which supports this reflection, in terms of the incident electric field amplitude phasor $\mathbf{E}_0$ at the origin.

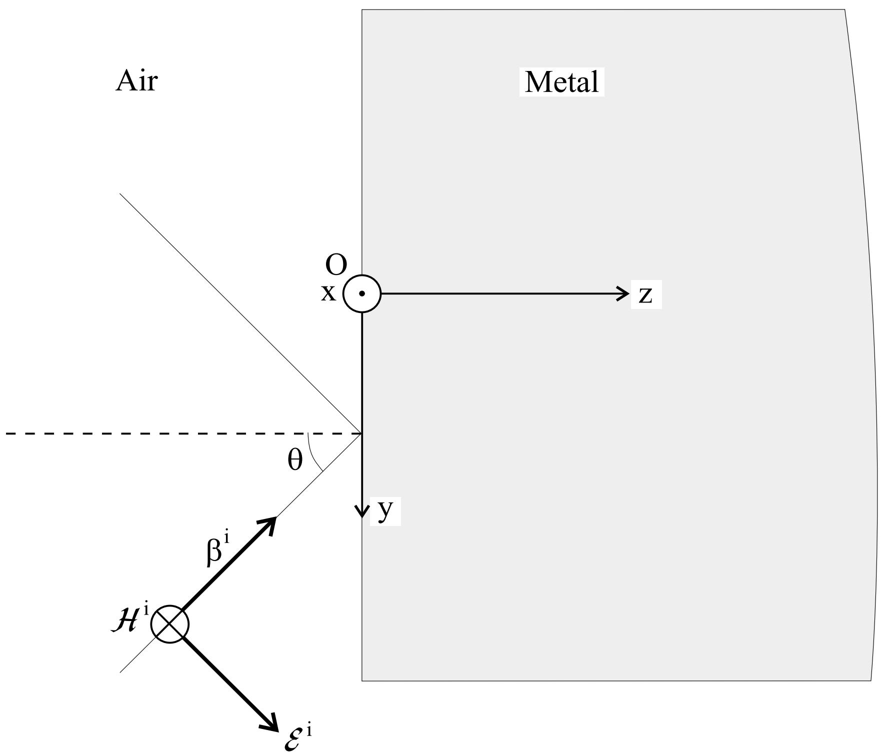


Figure K.9: Plane wave at oblique incidence on a perfect conductor.

K.12 Boundary Conditions

1. Defining carefully the notation introduced, state in integral form and in the time domain Ampere's law as modified by Maxwell. Interpret physically all terms of the

equation.

2. Using the above law, establish the boundary conditions which obtain for the tangential components of the magnetic field adjacent to but on opposite sides of the plane boundary between an insulating and a perfectly conducting medium.

K.13 Plane Waves

The electric field intensity associated with a plane electromagnetic wave incident in vacuum on the boundary $z = 0$ of a perfect conductor is given in SI units by the expression:

$$\mathsf{E}(x, y, z, t) = 10 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos(2\pi f t - z)\,\mathrm{Vm}^{-1}. \tag{K.2}$$

Determine:

1. An expression as a function of position and time for the magnetic field of the incident wave.

2. A numerical value for the frequency f of the wave.

3. A numerical value for the power density carried by the wave.

4. An expression as a function of position and time for the real surface current density set up by the wave at its reflection in the plane $z = 0$.

K.14 Radiation

1. (a) Calculate the radiation resistance of a short dipole of length 10 cm, carrying an oscillating current of 100 mA peak at 150 MHz, the current being assumed to be uniform along its length.
   (b) Calculate the total power radiated by the dipole.

2. If the dipole is located at the origin and is directed along the z axis, calculate the strength of the electric field radiated at a distance of 1000 m:
   (a) along the z axis; and
   (b) along the x axis.

3. Calculate the skin depth in copper (conductivity 5.8×10^7 S/m) at 150 MHz, and hence derive the series loss resistance of the above dipole assuming it has the form of a copper rod 2 mm in diameter. Compare this value with the radiation resistance.

4. It may be argued that the assumption of a uniform current distribution on the antenna is unrealistic. What do you consider would be a more reasonable assumption to make for the distribution of current, and how would the calculations of radiation resistance and loss resistance then be modified?

5. The calculations have necessarily ignored the question of the input reactance of the dipole. What form of input reactance (high, low, zero, capacitive, inductive) would you expect?

6. To what extent would matching of the dipole input impedance to the output impedance of commonly available generators affect the overall efficiency of the system as a radiator of electromagnetic energy?

K.15 Transmission Line Losses

1. Calculate the attenuation constant α in nepers per metre, and the transmission loss in dB per metre, for an air-filled co-axial line of inner and outer conductor radii a and b, respectively, in terms of conductor material conductivity σ and the proportions of the line.

2. How does this result scale with frequency?

3. What are the losses for an air-filled copper co-axial line of inner conductor radius 2 mm and outer conductor radius 7 mm at frequencies of:

   (a) 100 kHz; and

   (b) 10 Mhz?

4. What length of this cable would you use as a high power dummy load to achieve, at a frequency of 1.0 GHz, and for any value of load impedance, an input VSWR of less than or equal to 1.5?

K.16 More on Coaxial Line Losses

1. How do the losses of a co-axial line vary with Z_0, if the outer diameter is kept fixed?

2. Is there an optimum Z_0, and if so what is its approximate value, if the dielectric constant is 2.25?

K.17 Skin Depth and Waveguide Loss

1. Calculate the skin depth and surface resistivity in copper for frequencies of:

   (a) 1 MHz;

   (b) 100 Mhz; and

   (c) 1 Ghz.

2. Derive an expression for the attenuation constant for the dominant, i.e. TE_{10}, mode of a rectangular wave guide of height b and width a.

3. Calculate the attenuation, in dB/m, for copper waveguide at a frequency of 10 GHz.

K.18 Double Stub Tuner

A load of impedance $(11 - j4)\ \Omega$ is connected via a length of 0.34 wave lengths of transmission line of characteristic impedance 50 Ω to a double stub tuner, as shown in the Figure K.10.

The tuner consists of a length of 0.3 wave lengths of transmission line of characteristic impedance 70 Ω at each end of which is connected a variable length short circuited stub transmission line of characteristic impedance 100 Ω.

The tuner is intended to produce a match of the load to the long 50 Ω transmission line shown at the left of the figure. Determine stub lengths which will accomplish this result.

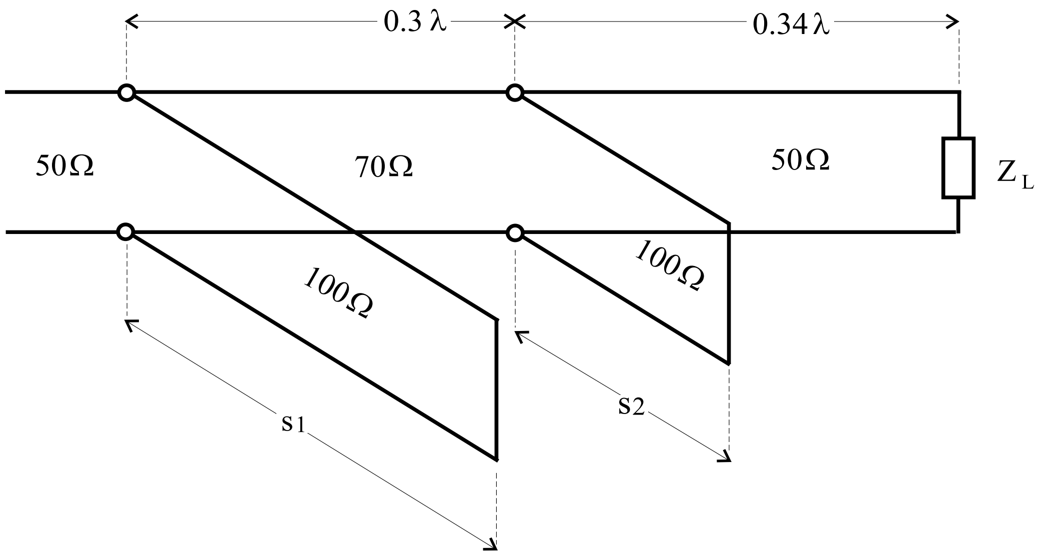


Figure K.10: A double stub tuner.

K.19 Transients on Transmisison Lines

In the circuit diagram below, the circuit S acts as a source of short duration transient signals, the transmissions system T contain two dissimilar cables, and the wideband oscilloscope C has an input impedance of 50 Ω, unaccompanied by any capacitance. The capacitor is initially charged to a voltage of 10V.

1. Neglecting the loading of the transmission lines, calculate the transient voltage developed across the 1 Ω resistor after the switch is closed.

2. Calculate the reflection and transmission factors where the 70 Ω line meets the 50 Ω line and where the 50 Ω load terminates the 70 Ω line.

3. Calculate the propagation times along the 4 m and 2 m lengths of line.

4. Sketch the form of voltage detected by the oscilloscope for the first 50 ns after the switch is closed.

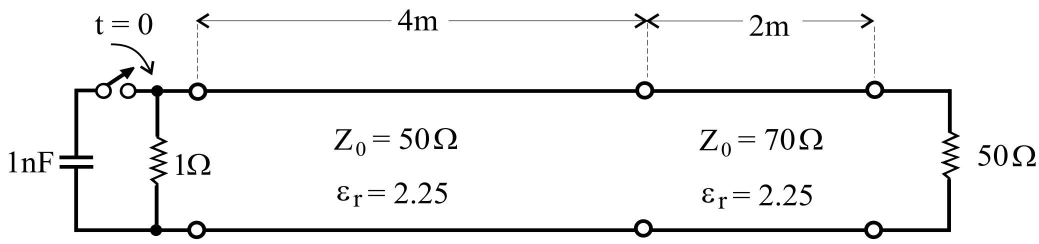


Figure K.11: Transmission lines feeding a wide band oscilloscope

K.20 An Old Friend

The parallel plate capacitor shown below is partly filled with a dielectric slab of dielectric constant 2.25, the remaining space being occupied by air. A voltage of 10 volts is established between the plates.

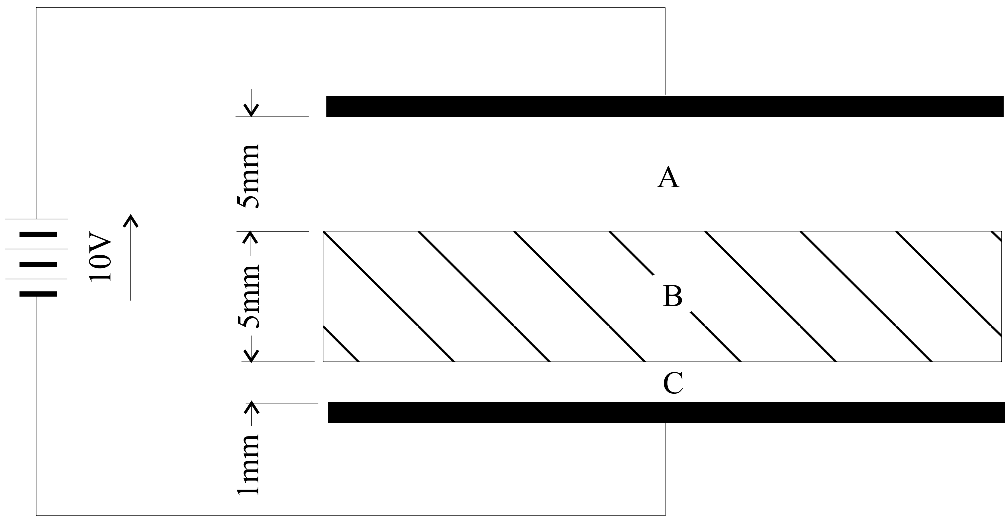


Figure K.12: Partially filled parallel plate capacitor.

Calculate in an appropriate order:

1. The electric field in the regions A, B and C.
2. The electric flux density in those regions.
3. The surface charge density on each of the plates.
4. The induced surface charge on each of the dielectric surfaces.

K.21 More Exercises on Notation and Plane Waves

1. Construct a vector phasor to represent the magnetic field which is expressed in SI units by the expression:

$$\mathsf{H}(x,y,z,t) = \begin{bmatrix} 0 \\ e^{-4z}\cos(1+1.2\times 10^9 t - 5y) \\ 0 \end{bmatrix}. \qquad \text{(K.3)}$$

2. Determine for this wave:

   (a) the frequency in Hz;

   (b) the complex propagation vector;

   (c) the planes of constant phase; and

   (d) the planes of constant amplitude.

3. For the above wave use the Maxwell equation giving the curl of the magnetic field to determine the vector phasor representing the corresponding electric field, assuming that the fields exist in empty space.

   For interest it may be stated that these fields are those which may be found in empty space adjacent to a plane boundary with a dielectric region in which is occurring total internal reflection of a uniform plane wave incident at a suitable angle on the boundary from within the dielectric.

4. Does the field described by the given equations itself correspond to a uniform plane wave? If so what is the direction of that wave? If not, why not?

K.22 Transmision Line Interpretation of Wave Reflection

A uniform plane transverse electromagnetic wave at a frequency of 3750 MHz is normally incident from the left upon a lossless dielectric slab, of thickness 10 mm and relative dielectric constant 4, which is backed by a perfectly conducting plate in the plane $z = 0$. If the wave is polarised with its electric field parallel to the x-axis, find the following:

1. The transmission line arrangement which is analogous to the slab and plate assembly.

2. The resultant reflection co-efficient at the air-dielectric interface.

3. Expressions for the resultant electric and magnetic field distributions in the free space region to the left of the air-dielectric interface.

4. Expressions for the resultant electric and magnetic field distributions in the dielectric.

5. Using the above results or otherwise determine the percentage of the incident power which is reflected by the structure.

K.23 Antennas

1. Determine the power receivd by a properly matched antenna which is distant 1000 m from a transmitter antenna radiating a power of 100 W under the following conditions:

   (a) Gain of transmitting antenna = 1.5.
   Effective area of receiver = 0.4 m^2.

   (b) Gain of both antennas = 2.
   Wavelength = 0.1 m.

   (c) Effective area of both antennas = 1 m^2.
   Wavelength = 0.03 m.

2. A small square loop antenna of side L centered at the origin and oriented with sides parallel to the x and y axes carries a sinusoidal current of which the phasor I represents the peak value.

   (a) Calculate the radiation vector $\mathbf{R}(\theta, 0)$ as a function of θ for directions in the x, z plane.

   (b) Locate the direction of strongest radiation.

   (c) Obtain expressions for the electric and magnetic field components in this direction.

Appendix L

ANSWERS TO EXERCISES

In this Appendix, the original questions are re-produced along with the answers. The section numbering used in this Apendix follows that used in Appendix K wherein the questions originally appeared.

L.1 Exercises on Notation

1. Construct phasors to represent:

   (a) $v(t) = 10\cos(\omega t + \pi/2)$ V

   (b) $i(t) = 5\sin\omega t$ A.

2. Find the real voltage as a function of position and time represented by the phasor $v(z) = \mathrm{V}_f e^{-j\beta z}$ where $\mathrm{V}_f = (3 + 3j)$ V.

Answers

1. (a)

$$\begin{aligned} \mathrm{V} &= 10e^{j\pi/2} \text{ V} \\ &= 10j \text{ V} \\ &= 10\angle 90^\circ \text{ V} \end{aligned}$$

   are suitable alternative forms.

   (b) We must first convert to a cosine function, using the relation $\sin(\omega t) = \cos(\omega t - \pi/2)$. Note that the relation $\sin(\omega t) = \cos(\pi/2 - \omega t)$ is not suitable because it reverses the sign of ωt. Thus $i(t) = 5\cos(\omega t - \pi/2)$ A. The phasor is then

$$\begin{aligned} \mathrm{I} &= 5e^{-j\pi/2} \text{ A} \\ &= -5j \text{ A} \\ &= 5\angle -90^\circ \text{ A} \end{aligned}$$

2. It is first convenient to convert V_f to polar form, ie $\mathrm{V}_f = 3\sqrt{2}e^{j\pi/4}$ V. Then $\mathrm{V}(z) = 3\sqrt{2}e^{-j\beta z + j\pi/4}$ V. Inserting a factor $e^{j\omega t}$ and taking the real part, we obtain the real voltage function

$$v(z,t) = 3\sqrt{2}\cos(\omega t - \beta z + \pi/4) \ \ \mathrm{V} \tag{L.1}$$

L.2 Transmission Line Fields

For the coaxial transmission line shown in Figure L.1, in which the centre conductor has dc current i, and steady charge q per unit length, use the integral forms of Gauss' and Ampere's laws to calculate the values of the electric field and the magnetic field at a radius r. Assume that the centre conductor is supported by a non-magnetic dielectric of dielectric permittivity ϵ.

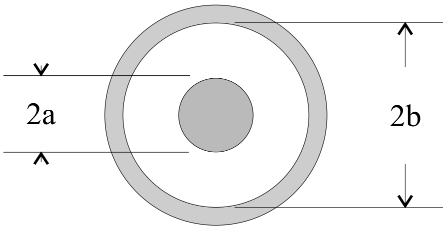


Figure L.1: Co-axial line configuration.

Answer

Somehow we seem to have set this question twice, once here and once in a later section. In a future revision of these notes, we will almost certainly preserve the present setting and delete the later one. We will for the present provide the outline of an answer here, and also in the later section. Is this a case of two answers for the price of one?

In respect of this question, the student could assume a short length δz of line, a line charge density q per unit length on the inner conductor, and a current i on that conductor. Then by employing Gauss' law of electrostatics, an expression for the radial component of electric flux density, and then the electric field, at a distance r from the centre can be derived. By employing Ampere's law of magnetostatics, an expression for the circumferential component of the magnetic field, and then the magnetic flux density, at a distance r from the centre can be derived.

Integration of the electric field expression, with due care to signs and the order of limits of the integral, should then give the potential of the inner conductor relative to the outer conductor, and should lead, after this result is divided into the charge per unit length, to an expression for the capacitance per unit length.

Calculation of the flux linked by a surface of length δz and width from the inner to outer conductor, and dividing by the current i, should give the inductance of the line. The results should be

$$C = \frac{2\pi\varepsilon}{\log_e\left(\frac{b}{a}\right)}$$

$$L = \frac{\mu_0}{2\pi} \log_e \left(\frac{b}{a}\right)$$

L.3 Transients on Transmission lines

1. A lossless transmission line, initially charged to a dc voltage V, is shorted at its input at $t = 0$. Sketch the voltage $v(t)$ at the open-circuit output end.

   Answer

   Let T be the time for a wave to propagate one way along the length L of the line. Before the switch is closed the forward and backward waves are $V_f(z,t) = \frac{1}{2}$ V, $V_r(z,t) = \frac{1}{2}$ V, so that everywhere on the line $v(z,t) =$ V.

   The discussion is slightly simplified by putting the switch on right, as:

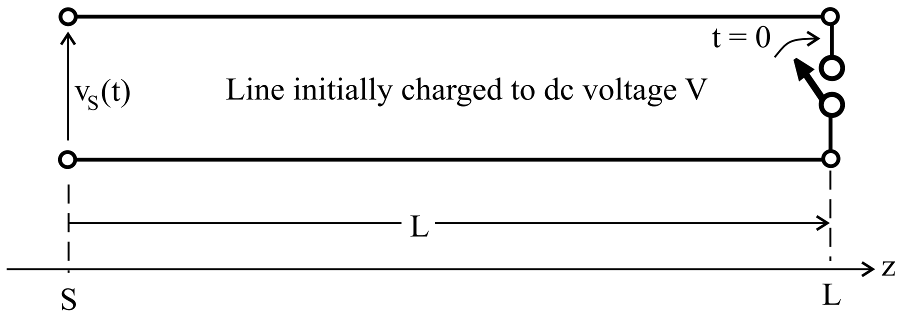


Figure L.2: Charged line with shorting switch.

   When the switch is closed $V_f(S,t)$ will initially remain the same, and so will $V_r(S,t)$ at least for $0 \leq t \leq T$. At the load end $V_f(L,t)$ remains the same initially, and $V_r(L,t)$ becomes $-\frac{1}{2}$V to produce zero total $v(L,t)$ at that end. After time T, the reverse wave of $-\frac{1}{2}$V arrives at $z = 0$. At this time $V_f(S,t)$ changes to $-\frac{1}{2}$V as well, because $V_f(S,t)$ may be considered as arising from $V_r(S,t)$ by reflection for the open circuit load end, and the reflection factor for an open circuit is $\Gamma_v = -1$. Continuing in this way we see the waveforms are as given in Figure L.3

2. A pulse generator consists of 10 metres of 50 Ω transmission line in which the velocity is 2×10^8 m/s, charged to 6 kV dc, and connected to a load impedance of 25 Ω by a relay which is closed at time $t = 0$. Sketch to definite scales the output pulse obtained. Calculate the peak output power and the total output energy in the pulse.

   Answer

   The propagation time T is 50 ns. The voltage reflection factors are at the sending end $\Gamma_v(S) = +1$ and at the load end $\Gamma_v(L) = -\frac{1}{3}$. Relations between V_f and V_r established at the ends of the line for $t > 0$ are at the sending end $V_f(S,t) = V_r(S,t)$ and at the load end $V_r(L,t) = -\frac{1}{3}V_f(L,t)$. Analysis of sucessive reflections gives the waveforms shown in Figure L.5.

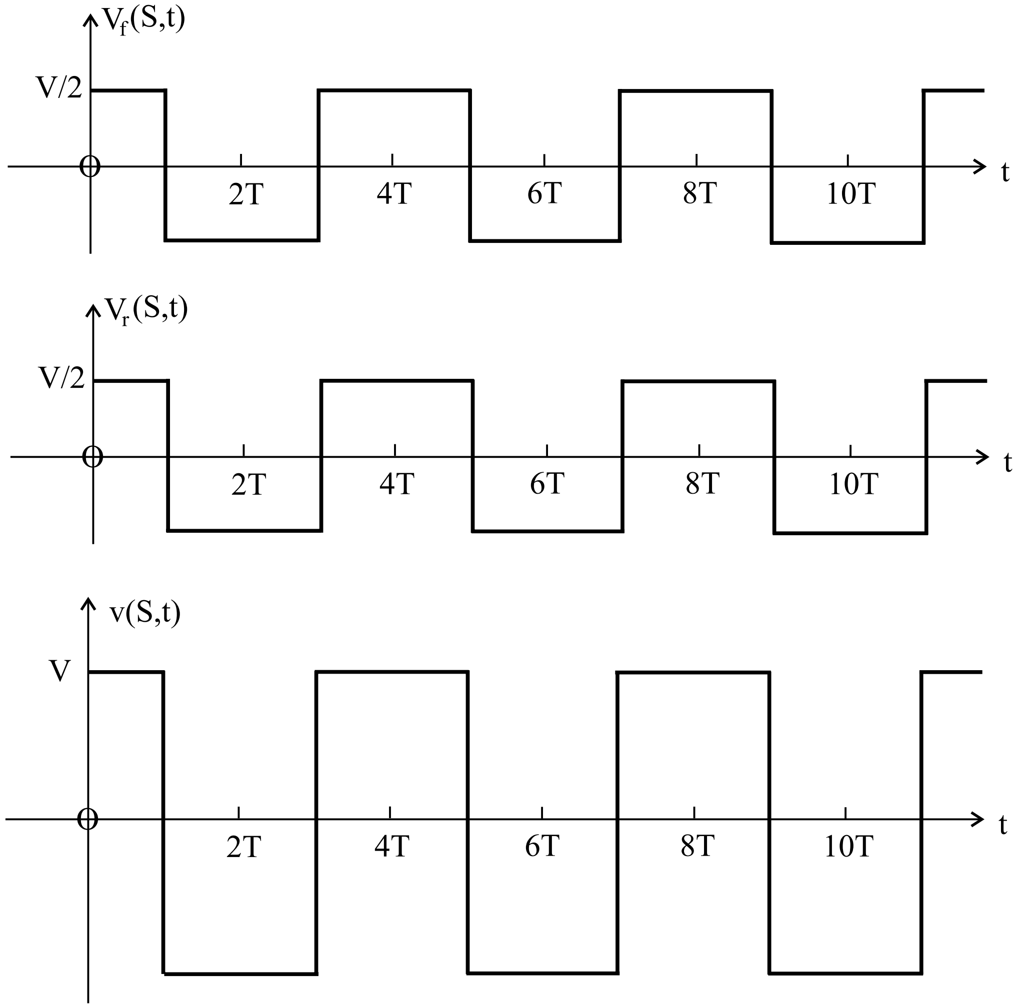


Figure L.3: Waveforms for charged line with shorting switch.

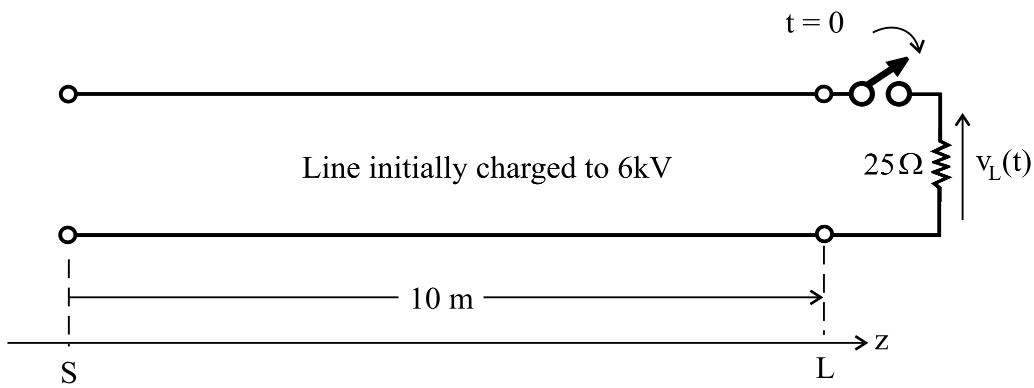


Figure L.4: Transmision line pulse generator.

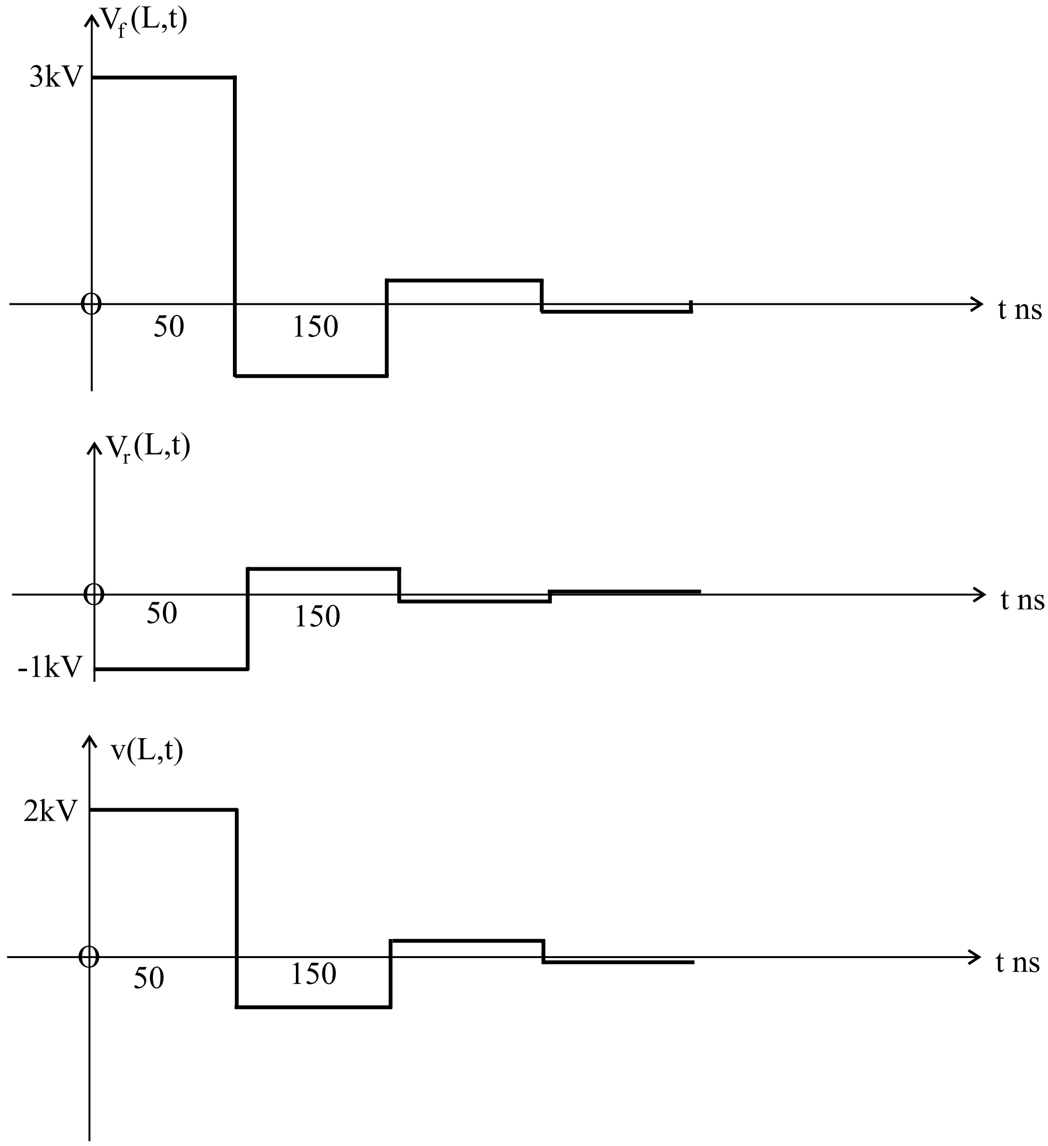


Figure L.5: Waveforms for transmision line pulse generator.

The peak output power, which occurs in the time interval $0 \leq t \leq 2T$, is given by

$$\begin{aligned} P_0 &= \frac{2000^2}{25} \\ &= 160 \text{ kW} \end{aligned}$$

The calculation of total output energy may be done in two ways. Firstly we may note that the output voltage in the nth time interval of length $2T$ is, using the notation V for the initial voltage of 6 kV on the line,

$$v_n = -V \left(\frac{-1}{3}\right)^n \tag{L.2}$$

the output energy in that interval is given by

$$W_n = \frac{V^2}{R_L} \left(\frac{1}{9}\right)^n 2T \tag{L.3}$$

The sum to infinity of these energies is

$$\begin{aligned} W_T &= \frac{V^2}{R_L} \left(\frac{1}{8}\right) 2T \\ &= \frac{V^2 T}{2Z_0} \end{aligned}$$

where we have used $R_L = Z_0/2$.

As an alternative, we may note that the total output energy is the energy stored initially in the line, and that if R_L were changed to Z_0 this energy would be simply output in the form of a single pulse of amplitude $V/2$ and duration $2T$ across a resistor Z_0. The total output energy in such a pulse is

$$\begin{aligned} W_T &= \left(\frac{V}{2}\right)^2 \frac{2T}{Z_0} \\ &= \frac{V^2 T}{2Z_0} \end{aligned}$$

as before. With the numerical values given we find that $W_T = 18$ mJ.

3. A d.c. voltage source of internal voltage 1200 volts and internal resistance 10 Ω is connected at $t = 0$ to 10 metres of lossless co-axial transmission line. Measurements at low frequencies show that the length of line has a total capacitance of 1.0 nF and a total inductance of 2.5 μH. The line is terminated in load resistance of 30 Ω.

(a) Determine the characteristic impedance and the wave velocity for the coaxial line.

Answer

The inductance and capacitance per unit length are 250 nH/m and 100 pF/m. Hence

$$\begin{aligned} Z_0 &= \sqrt{\tfrac{L}{C}} &= 50\,\Omega \ \text{, and} \\ c &= \tfrac{1}{\sqrt{LC}} &= 200 \text{ Mm/s} \end{aligned}$$

(b) Draw a schematic circuit diagram and below it a lattice diagram of the forward and reverse waves of voltage.

Answer

(c) Determine the transit time for a wave front to travel from the source to load and the reflection coefficients for each end of the line, and show them on the lattice diagram.

Answer

Transit time $\tau = \dfrac{10 \text{ m}}{200 \text{ Mm/s}} = 50$ ns

$$\begin{aligned} \Gamma_L &= \frac{30-50}{30+50} = -\frac{1}{4} \\ \Gamma_S &= \frac{10-50}{10+50} = -\frac{2}{3} \end{aligned}$$

These values are shown on the lattice diagram of Figure L.6.

(d) Show details of the voltage distribution as a function of position along the line at 1.5 and 2.5 times the one way transit time.

Answer

At 1.5 times the transit time we have a forward wave of 1000 V has reached the load and a reflected wave of -250 V has reached half way back to the source. At 2.5 times the transit time we have the -250 V has reached the source and produced and additional forward wave of 167V,which has travelled half way to the load. Hence the diagrams shown in Figure L.7.

(e) Show details of the voltage waveform as a fuction of time at the load terminals.

Answer

At the load terminals we have a voltage

$$\begin{aligned} & V_S \frac{Z_0}{Z_S + Z_0}(1+\Gamma_L)u(t-T) = 1000V\left(\frac{3}{4}\right)u(t-T) \\ + \quad & V_S \frac{Z_0}{Z_S + Z_L}(1+\Gamma_L)\Gamma_S\Gamma_L u(t-3T) = 1000V\frac{3}{4}\frac{1}{6}u(t-3T) \\ + \quad & \text{more terms with an extra factor of } \frac{1}{6} \text{ each time.} \end{aligned}$$

A diagram of the waveform appears in Figure L.8.

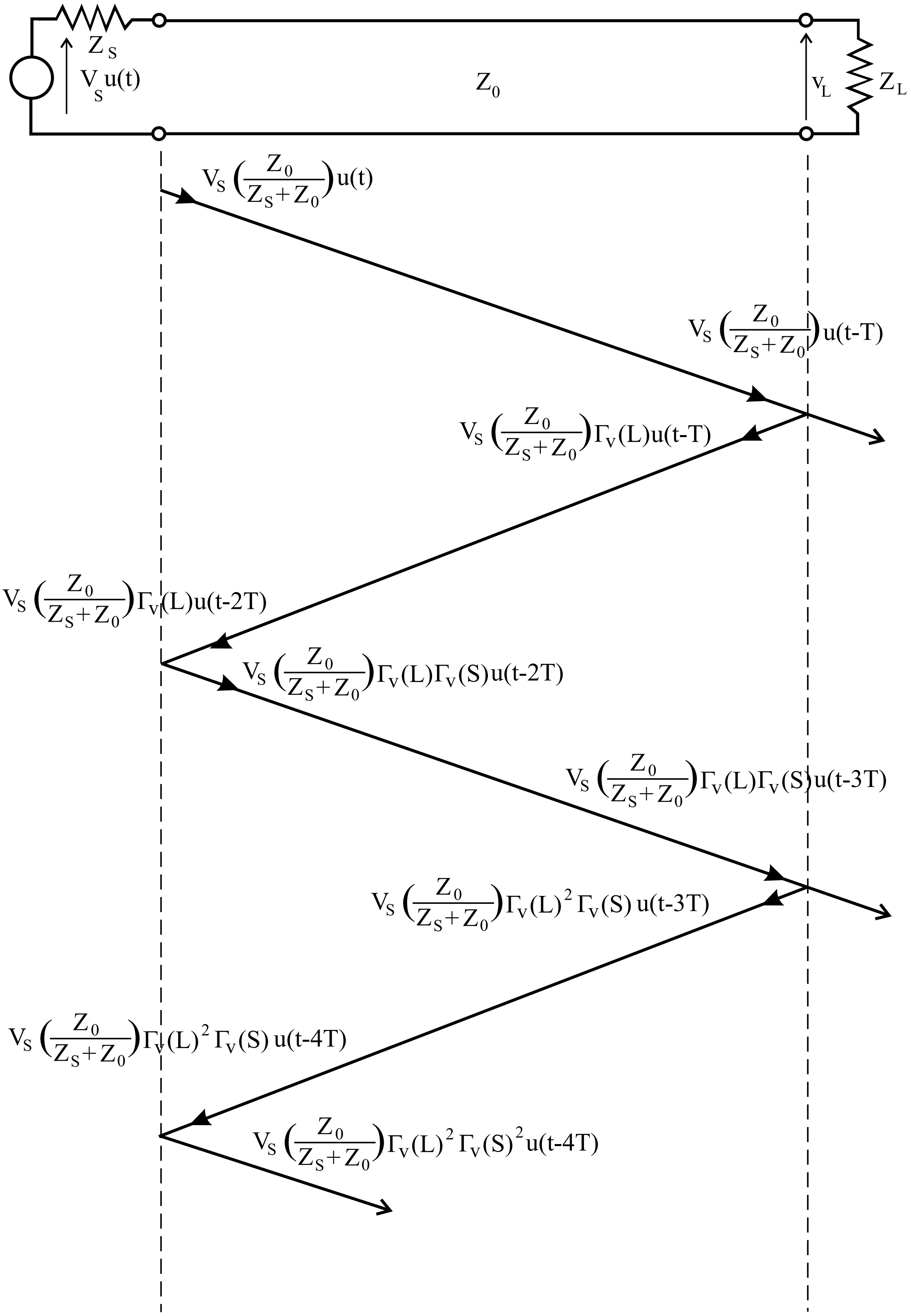


Figure L.6: Lattice diagram for the coaxial transmission line.

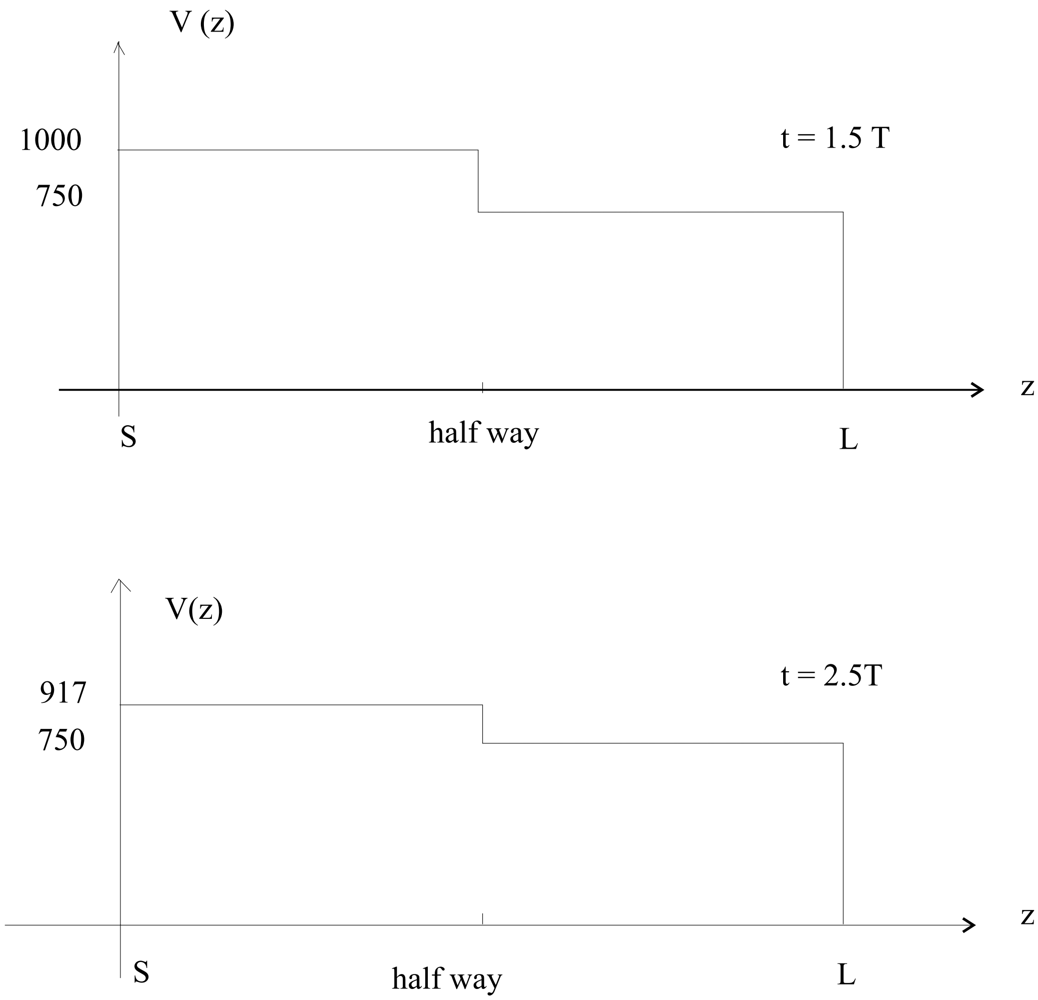


Figure L.7: Total voltage at times 1.5T and 2.5T

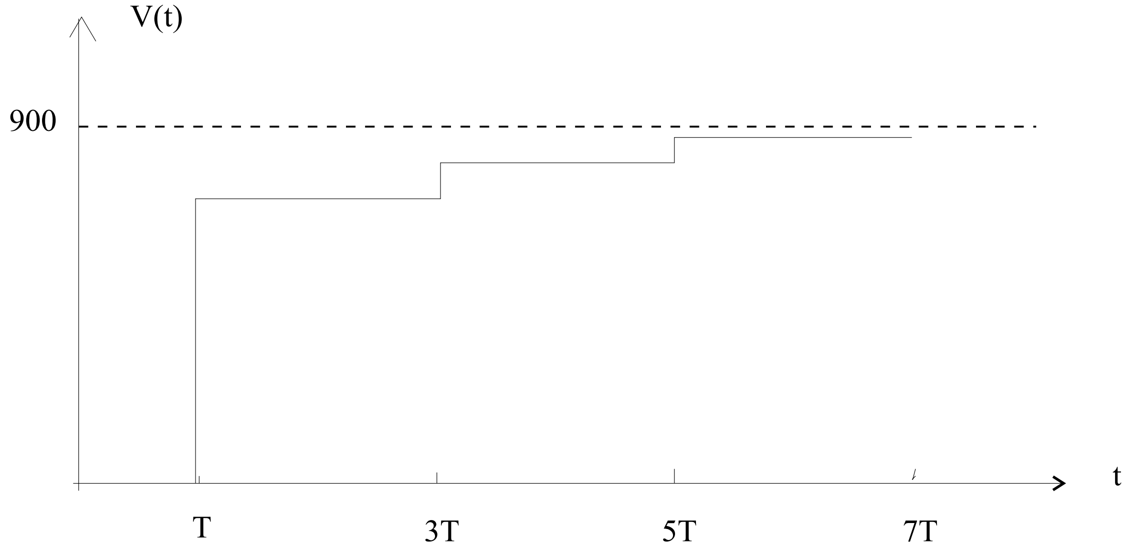


Figure L.8: Load end voltage as a function of time.

(f) What is the steady state voltage distribution for this line with the source and load specified?

Answer

Eventually the line takes a steady state voltage of $V_S \frac{Z_L}{Z_S + Z_L} = 1200 \frac{30}{10 + 30} = 900V$.

L.4 Transmisson Lines in the Frequency Domain

1. Calculate the inductance per unit length of an air-cored co-axial line of radii a (inner conductor) and b (outer conductor).

   Answer

   In respect of this question, the student could assume a short length δz of line, a line charge density q per unit length on the inner conductor, and a current i on that conductor. Then by employing Gauss' law of electrostatics, an expression for the radial component of electric flux density, and then the electric field, at a distance r from the centre can be derived. By employing Ampere's law of magnetostatics, an expression for the circumferential component of the magnetic field, and then the magnetic flux density, at a distance r from the center can be derived.

   Integration of the electric field expression, with due care to signs and the order of limits of the integral, should then give the potential of the inner conductor relative to the outer conductor, and should lead, after this result is divided into the charge per unit length, to an expression for the capacitance per unit length.

   Calculation of the flux linked by a surface of length δz and width from the inner to outer conductor, and dividing by the current i, should give the inductance of the line. The results should be

$$
\begin{aligned}
C &= \frac{2\pi\varepsilon}{\log_e\left(\frac{b}{a}\right)} \\
L &= \frac{\mu_0}{2\pi}\log_e\left(\frac{b}{a}\right)
\end{aligned}
$$

2. What is the input impedance of each of the lossless lines shown in Figure L.9?

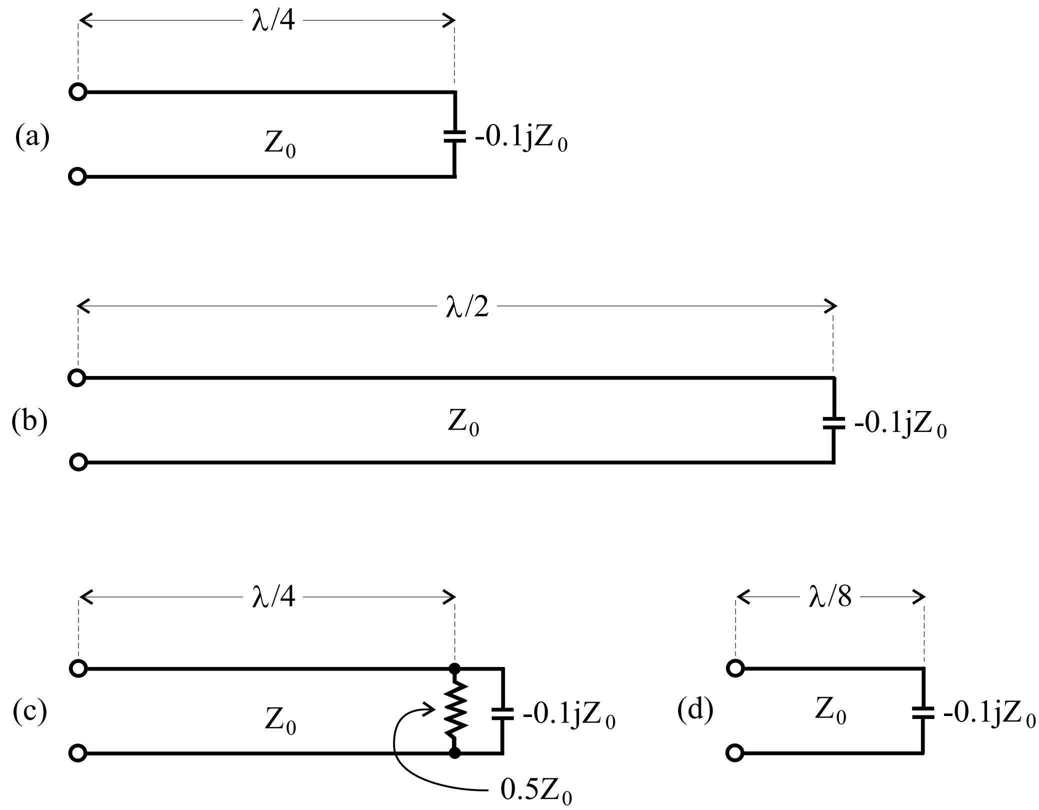


Figure L.9: Lossless transmission lines.

Answer

The answers to the first three parts of this question may be easily obtained from the facts that a quarter wave length of line transforms a load impedance Z_L into an input impedance of $Z_I = Z_0^2/Z_L$, while a half wave length of line transforms a load impedance Z_L into itself. The answer to the last question is obtained by using the equation

$$
\frac{Z_I}{Z_0} = \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} \tag{L.4}
$$

and noting that both $\cos\beta l$ and $\sin\beta l$ are equal to $1/\sqrt{2}$. The results are:

(a) $Z_i = 10jZ_0$;

(b) $Z_i = -0.1jZ_0$;

(c) $Z_i = (2 + j10)Z_0$; and

(d) $Z_i = 0.818jZ_0$.

3. A transmission line has the following distributed parameters per unit length: $L = 0.5\ \mu\text{H/m}$, $R = 2.0\ \Omega/\text{m}$, $C = 50\ \text{pF/m}$, and $G = 0\ \text{S/m}$.

   Calculate at a frequency of 31.8 MHz the characteristic impedance, and the attenuation and phase constants.

   Answer

   At 31.8 Mhz, $\omega L = 100\Omega/\text{m}$ and $\omega C = 0.01\text{S/m}$. Hence the characteristic impedance is

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{2 + j100}{j0.01}} = 100\sqrt{1 - 0.02j} \tag{L.5}$$

   Using a two-term binomial expansion we obtain

$$Z_0 = (100 - j1)\Omega \tag{L.6}$$

   The complex propagation is

$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{(2 + j100)(j0.01)} \\ &= j\sqrt{1 - j0.02} \\ &\approx j + 0.01 \end{aligned}$$

   Hence the attenuation constant is 0.01 neper per metre and the phase constant is 1 radian per metre.

4. A *distortionless line* is defined as one in which α is independent of frequency and β is proportional to frequency. Show that this condition occurs when $R/L = G/C$.

   Answer

   From the basic propagation equations we have $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$. Under the given conditions this becomes

$$\begin{aligned} \alpha + j\beta &= j\omega\sqrt{LC\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)} \\ &= j\omega\sqrt{LC\left(1 + \frac{R}{j\omega L}\right)^2} \\ &= \left(j\omega + \frac{R}{L}\right)\sqrt{LC} \end{aligned}$$

Hence $\beta = \omega\sqrt{LC}$ and $\alpha = R\sqrt{\frac{C}{L}}$, i.e. α is independent of frequency and β is proportional to frequency. Thus all frequency components of a complicated waveform will travel with the same velocity, and all will suffer the same attenuation with distance. These features ensure that the wave propagates without distortion.

5. Calculate the input impedance of 1000 m of the transmission line described in Problem 3. Explain why this does not depend on the load impedance Z_L.

Answer

We observe that for the line in Problem 3 above, the product of the attenuation constant α and twice the line length is 20. Thus any forward wave launched at the sending end will, even if fully reflected at the load end, suffer attenuation by a factor of e^{20}, ie by about 174 dB, before it gets back to the input. Hence at the input, only V_f and I_f are present to significant amplitude. Hence the input impedance is

$$Z_{in} = \frac{\mathrm{V}_{in}}{\mathrm{I}_{in}} = \frac{\mathrm{V}_f}{\mathrm{I}_f} = Z_0 \tag{L.7}$$

6. An impedance of $(100 + \jmath 100)\ \Omega$ is placed as a load on a lossless transmission line of characteristic impedance 50 Ω. Find the reflection coefficient in magnitude and phase at the load end. What is its magnitude and phase at the input end if the line is $3\lambda/8$ long at the operating frequency? How does this impedance change as the frequency varies?

Answer

The voltage reflection factor is given by

$$\begin{aligned}\Gamma_v(L) &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= \frac{50 + \jmath 100}{150 + \jmath 100} \\ &= \frac{112\angle 63.4^\circ}{180\angle 33.7^\circ} \\ &= 0.620\angle 29.7^\circ\end{aligned}$$

The voltage reflection factor at the input is the same in magnitude as at the outupt but is retarded in phase by 720° for each wave length of line. For a $3\lambda/8$ line, the phase retardation is 270°. Thus

$$\Gamma_v(S) = 0.620\angle 119.7^\circ$$

As the frequency is varied, we will assume in the absence of other information that Z_L will not change (a somewhat dubious assumption). Then $\Gamma_v(L)$ will not change, and $\Gamma_v(S)$ will change in phase, (but not in magnitude), becoming further retarded in phase as the line length increases. Whatever happens to $\Gamma_v(L)$, we can be sure that when the frequency is raised so that the line becomes a half wave in length, the input reflection factor becomes equal to the load reflection factor.

Some sensible discussion along these lines, but not necessarily incorporating all the points above, should attract full marks.

7. In the circuit in Figure L.10, the transmission line has air dielectric, and the resistors are lumped elements.

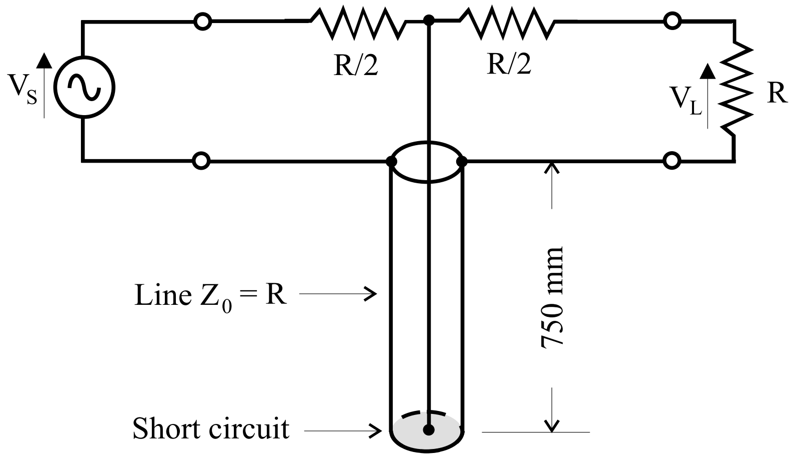


Figure L.10: Transmission line in a lumped parameter circuit.

(a) Sketch the frequency response of the circuit over the frequency range from 0 to 500 MHz.

(b) Sketch the frequency response when the short circuit on the transmission line is replaced by an open circuit.

Answer

(a) The shorted end transmission line will appear at its input end as
(a) a s/c at dc;
(b) an o/c at 100 MHz;
(c) a s/c at 200 MHz;
(d) an o/c at 300 MHz;
(e) a s/c at 400 Mhz; and
(f) an o/c at 500 MHz.
The approximate frequency response is shown in Figure L.11.

(b) The open circuited end transmission line will appear at its input end as
(a) an o/c at dc;
(b) a s/c at 100 MHz;
(c) an o/c at 200 MHz;
(d) a s/c at 300 MHz;

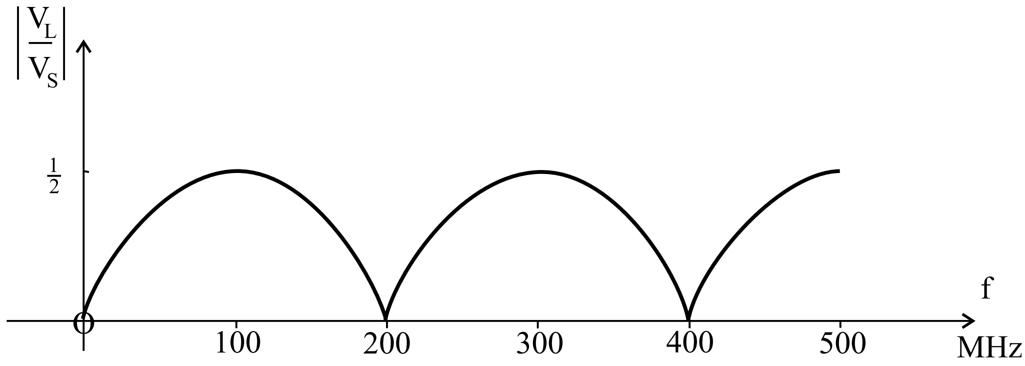


Figure L.11: Frequency response for circuit with shorted line.

(e) an o/c at 400 Mhz; and
(f) a s/c at 500 MHz.
The approximate frequency response is shown in Figure L.12.

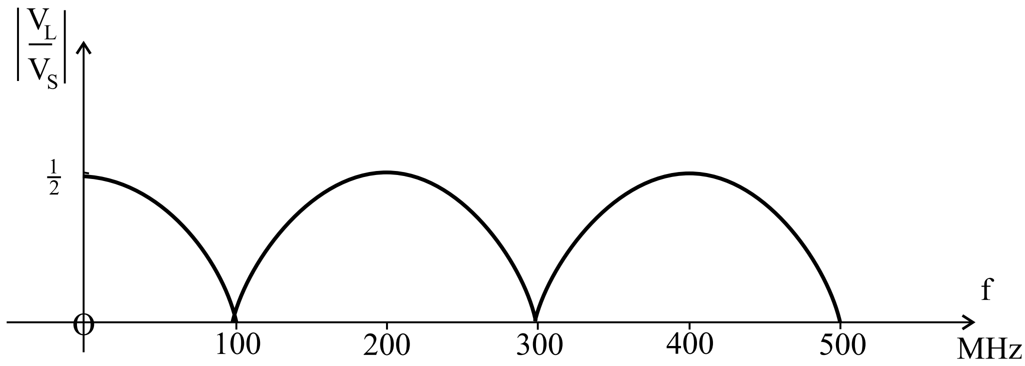


Figure L.12: Frequency response for circuit with open circuit line.

8. (a) At a frequency of 1000 MHz, to what capacitance is 10 cm of 50 Ω line, shorted at its end, equivalent? Assume that the dielectric is air.
   (b) If the line has instead a dielectric with dielectric constant 2.25, what is the capacitance?

Answer

(a) For an air line, $c = 3 \times 10^8$m/s. Hence $\beta = \omega/c = (20\pi/3)\text{m}^{-1}$ at 1 GHz. Hence $\beta l = 2\pi/3$ for $l = 0.1$m. For a shorted line, $Z_I = jZ_0 \tan(\beta l) = -86.6j\Omega$ when $\beta l = 2\pi/3$. The corresponding capacitance is given by

$$\begin{aligned} C &= \frac{1}{86.6\omega} \\ &= \frac{1}{86.6 \times 2\pi \times 10^9} \\ &= 1.84 \ \text{pF} \end{aligned}$$

(b) If the dielectric constant is 2.25, the capacitance per unit length is increased by this factor. The following changes result in the line parameters:

(i) β is increased by a factor $\sqrt{2.25} = 1.5$; and

(ii) Z_0 is decreasd by a factor $\sqrt{2.25} = 1.5$.

With the new value of β and the old line length of 0.1m, the product βl now has become π, i.e. the line is now half a wave length long. The input impedance is thus equal to the load impedance, ie a short circuit, and the equivalent capacitance becomes infinite.

L.5 Miscellaneous Transmision Line Problems

1. What is the input impedance at a single frequency for the transmission lines shown in Figure L.13, expressed as a large or small inductance or capacitance? In Figure L.13, δ is a length very much less than the wave length λ on the line at the frequency of interest.

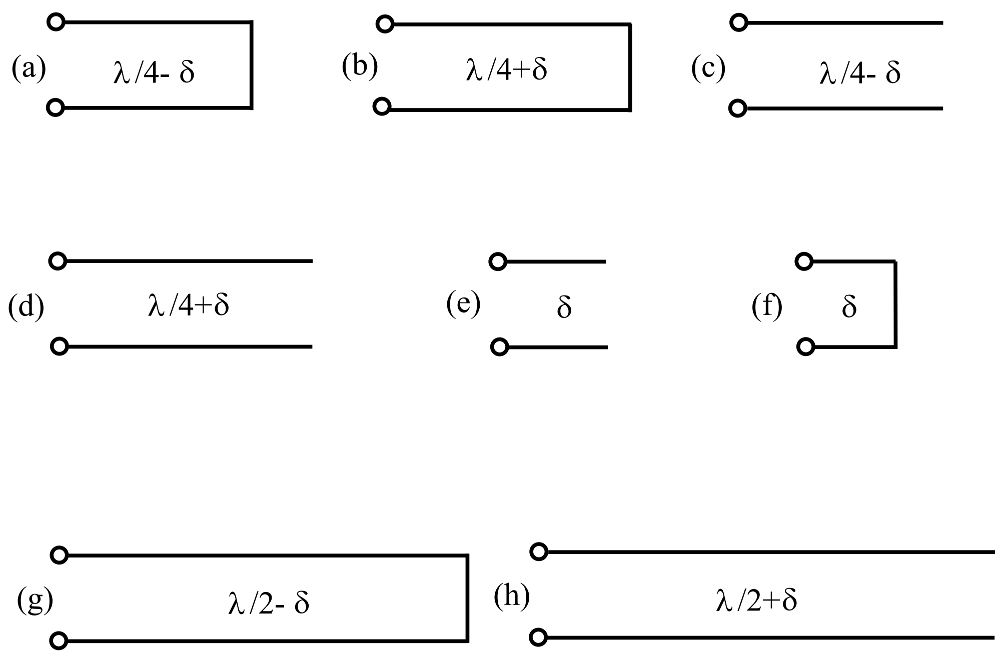


Figure L.13: Various open and short circuit transmisison lines.

Answer

For the shorted lines, we make use of the relation $Z_I = jZ_0 \tan(\beta l)$, which tells us that a shorted line of length much less than a quarter wave length is inductive and a low impedance, and that as the length increases it remains inductive but of increasing impedance until the length reaches a quarter wavelength, after which it becomes capacitive but initially of a high impedance, the impedance decreasing but remaining capacitive until a half wave length is reached at which point the impedance has reduced to zero. For lines of length greater than a half wave lenth, the behaviour described above repeats, ie we can take any integral number of half wave lengths off the line length.

For the open circuit lines, we make use of the relation $Z_I = -jZ_0 \cot(\beta l)$, which tells us that an open circuit line of length much less than a quarter wave length is capacitive and a high impedance, and that as the length increases it remains capacitive but of decreasing impedance until the length reaches a quarter wavelength, after which it becomes inductive but initially of a low impedance, the impedance increasing but remaining inductive until a half wave length is reached at which point the input has become an open circuit. For lines of length greater than a half wave lenth, the behaviour described above repeats, ie we can take any integral number of half wave lengths off the line length.

Using these insights, and noting that a small capacitance has a large impedance, we see that the illustrated lines behave as:

(a) a large inductance;

(b) a small capacitance;

(c) a large capacitance;

(d) a small inductance;

(e) a small capacitance;

(f) a small inductance;

(g) a large capacitance; and

(h) a small capacitance.

2. A transmission line of $Z_0 = 50\ \Omega$ is terminated in $Z_L = (100 + j100)\ \Omega$. Find the VSWR on the line, and the location of the first voltage minimum.

Answer

The normalized z_L is 2 + j2. Hence $\Gamma_v(L)$ is given by

$$\begin{aligned} \Gamma_v(L) &= \frac{z_L - 1}{z_L + 1} \\ &= \frac{1 + j2}{3 + j2} \\ &= 0.620\angle 29.7^\circ \end{aligned}$$

Hence

$$\begin{aligned} S &= \frac{1 + 0.620}{1 - 0.620} \\ &= 4.26 \end{aligned}$$

As noted above the phase of Γ_v at the load end is 29.7°. As we move away from the load, the phase of Γ_v is retarded, by an amount of 720° for each wave length. At a voltage minimum on the line, the phase of Γ_v is -180°. Thus we need to move $\frac{209.7}{720} = 0.291$ of a wave length back from the load to get to the first voltage minimum.

3. If a twin wire balanced telephone line is required to have a characteristic impedance of 600 Ω, calculate the required separation between conductors if enamel insulated copper wire 1 mm diameter is to be used.

Answer

Using the approximate relation for $s \gg d$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\log(2s/d)}{\pi}$$

with $Z_0 = 600\,\Omega$ and $\sqrt{\mu_0/\epsilon_0} = 120\pi\,\Omega$, we have

$$\begin{aligned} \frac{2s}{d} &= \exp\left(\frac{600}{120}\right) \\ &= 148.4 \end{aligned}$$

Thus $s = \frac{148.4}{2} \times 1$ mm $= 74.2$ mm.

4. A length of 80.62 cm of 50 Ω coaxial cable, in which the velocity is 2×10^8 m/s, is being used as a measurement cable at a frequency of 50 MHz on an oscilloscope which has an input capacitance of 20 pF, and negligible input conductance. Calculate the input impedance of the free end of the cable.

Answer

We can use the relation for input impedance

$$\frac{Z_I}{Z_0} = \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l}$$

where $Z_L = -j/(\omega C) = -j159.2\,\Omega$, $Z_0 = 50\,\Omega$, and $\beta = \omega/c = \pi/2$ m^{-1}.

Hence $\cos\beta l = .2997$ and $\sin\beta l = .9540$. Thus

$$\begin{aligned} Z_I &= \frac{-j159.2 \times .2997 + j50 \times .9540}{50 \times .2997 + 159.2 \times .9540} 50 \\ &= \frac{-j47.7 + j47.7}{15.0 + 151.9} 50 \\ &= 0. \end{aligned}$$

We see that to the accuracy to which these calculations have been made, the input impedance appears to be a short circuit. What has happened is that as the transmission line termination is lossless, the magnitude of the reflected wave is equal to the magnitude of the incident wave. For some line length, in fact for the line length chosen in this problem, the forward and reflected waves at the input will be exactly out of phase and will cancel to zero.

We should not however conclude that the oscilloscope will display no signal. If the oscilloscope is being used to determine signals in a circuit, that circuit at its point of connection to the oscilloscope cable can be represented by a Thevenin or Norton circuit. We can from this circuit calculate the current which will enter the line at its input. There will be forward and backward voltage and current waves on the line as a result. Although the voltage waves cancel at the input end, they will not cancel at the output end; in fact noting that the line in this example is only a little short of a quarter wave length, we can see that the forward and backward voltage waves will be approximately in phase at the output, and the voltage on the line will be close to its maximum value. If the internal impedance of the Thevenin or Norton circuit were small in relation to 50 Ω, quite a large current will enter the line, and a large voltage, in fact much larger that the source voltage of the equivalent circuit, will be displayed on the oscilloscope.

So we have seen that at high frequencies, measurements can be tricky, but we are grown people, and ought to be able to handle this fact. In fact figuring out what is going on can be fun.

5. A 50 Ω line is terminated in an impedance of $(75 - \jmath 69)\ \Omega$. The line is 3.5 m long and is excited by a source of energy at 50 MHz. The velocity of propagation is 3×10^8 m/s. Use the Smith chart to find the input impedance, the magnitude and phase of the input reflection coefficient, the standing wave ratio on the line, and the position of a voltage minimum.

   Answer

   The wave length of the line is $\frac{3\times10^8}{5\times10^7} = 6$ m. Hence the line length is 0.5833 λ. The normalised load impedance is 1.5 - $\jmath$1.38. From the Smith chart we find the normalised input impedance is 0.48 - $\jmath$0.69, and the input reflection coefficent is $0.493\angle-102°$. The VSWR is found to be 3.2, and the voltage minimum is found to be 0.192 λ, i.e. 1.15 m from the load.

6. The standing wave ratio on an ideal 70 Ω line is measured as 3.2, and a voltage minimum is observed 0.23 λ in front of the load. Use the Smith chart to find the load impedance.

   Answer

   After entering the Smith chart at a point defind $z = \frac{1}{3.2} + \jmath 0$ on the real axis, i.e the point corresponding to a voltage minimum of the standing wave pattern, and rotating by an angle corresponding to 0.23 λ i.e to-ward the load, i.e. counterclockwise, we find that the normalised load impedance is 2.8 - $\jmath$1. Hence

$$\begin{aligned} Z_L &= 70(2.8 - \jmath 1.0) \\ &= 196 - \jmath 70\,\Omega \end{aligned}$$

L.6 Transmission Line Matching

1. What current flows in the short circuit at the end of the lossless line shown in Figure L.14?

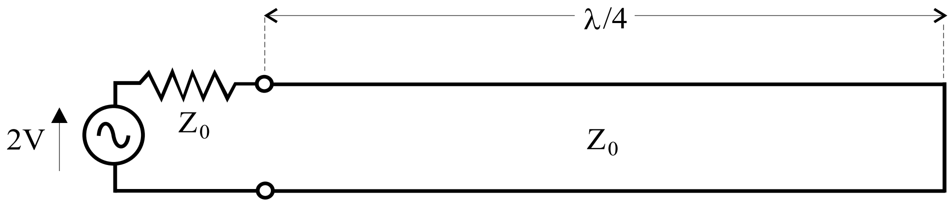


Figure L.14: Shorted lossless transmision line with matched source.

Answer

The line appears as an open circuit at the input. Hence a voltage of 2V appears at the input. Thus, at the input the forward and backward waves V_f and V_r are in phase and each equal to V. A the load end, V_f will be retarded in phase by $\pi/2$ ie will be $-j$V, and V_r will be advanced in phase by $\pi/2$ ie it will be jV. The current I will be $(\ \mathrm{V}_f -\ \mathrm{V}_r)/Z_0 = -2j\ \mathrm{V}/Z_0$. Note that this current will be obtained for any value of resistor in series with the generator, not merely the value shown.

2. If a coaxial cable with loss 0.2 dB per metre is used as in Figure L.15, what is the VSWR:

   (a) at the load end; and

   (b) at the source?

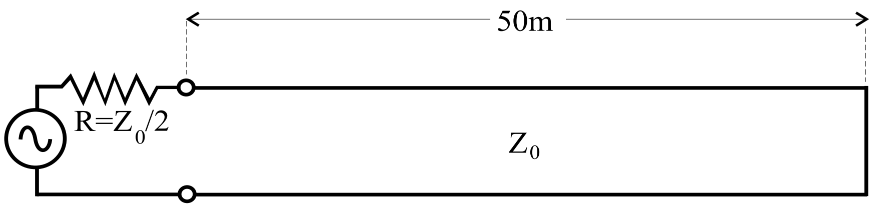


Figure L.15: Shorted lossy transmision line with unmatched source.

Answer

At the load end of the line we have $\Gamma_v = -1$ and $|\Gamma_v| = 1$. The VSWR is infinite. At the source end, we can assert that any forward wave sent down the line will arrive back attenuated by 20 dB. Thus at the source end $|\ \mathrm{V}_r| = 0.1|\ \mathrm{V}_f|$. The VSWR is then

$$\begin{aligned} S &= \frac{|\ \mathrm{V}_f| + |\ \mathrm{V}_r|}{|\ \mathrm{V}_f| - |\ \mathrm{V}_r|} \\ &= 1.22 \end{aligned}$$

3. Using single stub matching, how would you match a transistor, the output impedance of which is $(20 - j30)\ \Omega$, to an antenna of impedance $(50 + j20)\ \Omega$ via a long length of low loss 50 Ω cable, at a frequency of 300 MHz?

   Since the line is long, we must match both the transistor and the antenna separately to the line to get efficient power transfer. We will assume in the calculation of stub lengths that the cable has air dielectric and hence $\lambda = 1$ m. We will also assume shunt connected short circuited stubs. We solve each matching problem by first calculating the normalised impedance, and transforming to normalised admittance by reflection in the origin of the Smith chart. We regard the transistor and the antenna as load admittances in turn in the standard single stub matching procedure. The results of these calculations are in summary:

SUMMARY OF RESULTS			
Match to transistor		Match to antenna	
z_L	0.4 -j 0.6	z_L	1 + j 0.4
y_L	0.78 + j 1.15	y_L	0.86 - j0.34
stub position	0.016 λ 16 mm	stub position	0.250 λ 250 mm
y_s	-j1.33	y_s	-j0.4
stub length	0.102 λ 102 mm	stub length	0.19 λ 190 mm

Table L.1: Summary of results for transistor and antenna matching

4. A 50 Ω transmission line is terminated with a load $Z_L = (20 + j30)\ \Omega$. A double stub tuner consisting of a pair of shorted 50 Ω transmission lines connected in shunt with the main line and spaced 0.25 λ apart is located with one stub 0.2 λ from the load. Find the lengths of the stubs to give unity VSWR at 0.55 λ from the load, i.e. on the generator side of the tuner.

   Answer

   We simply are required to match the load to the line; unity VSWR will then result at all points on the line greater than 0.45 λ from the load. The procedure in brief is:

   - calculate the normalised z_L;
   - enter the Smith chart at z_L and reflect in the origin to obtain y_L;
   - transform 0.2 λ towards the generator; and
   - apply double stub matching as explained in lectures, making allowance for a stub separation of $\lambda/4$ instead of $3\lambda/8$ as normal.

   Again we assume shunt connected short circuited stubs of the same characteristic impedance as the main line. The results of the calculation are:

   (a) The length of the stub nearer the load is 0.282 λ.

(b) The length of stub nearer the generator is 0.407 λ

5. A 75 Ω line is terminated in $(50 + j100)$ Ω.

   (a) What is the VSWR? What percentage of forward power is reflected from the load?

   (b) What is the distance from the load back to the first voltage minimum, and to the first voltage maximum?

   (c) If the line is to be matched by a single stub, what is its length and position? Both the line and the stub have characteristic impedance $Z_0 = 75$ Ω.

   Answer

   (a)

$$\begin{aligned} z_L &= 0.67 + j1.33 \\ |\Gamma_v| &= 0.643 \\ \text{VSWR} &= 4.6 \\ \text{\% of reflected power} &= 41\% \end{aligned}$$

   (b)

$$\begin{aligned} \text{Distance to } V_{min} &= 0.34\ \lambda \ \text{ from load} \\ \text{Distance to } V_{max} &= 0.09\ \lambda \ \text{ from load} \end{aligned}$$

   (c)

$$\begin{aligned} \text{Stub length} &= 0.084\ \lambda \\ \text{Stub position} &= 0.27\ \lambda \text{ from load} \end{aligned}$$

6. In the matching system shown in Figure L.16, l_1 and l_2 are line lengths and s_1 and s_2 are the lengths of the stubs.

   (a) If $l_2 = 0$ and $l_1 = 3\lambda/8$, what values of R_L cannot be matched?

   (b) If $l_2 = 0$ and $l_1 = \lambda/4$, what values of R_L cannot be matched?

   (c) If $l_2 = \lambda/10$ and $l_1 = 3\lambda/8$, and $R_L = Z_0/2$ and $X_L = Z_0$, what stub lengths s_1 and s_2 are required for a match?

   (d) What is then the VSWR in line l_2, and what is it in line l_1?

   Answer

   (a) The values of R_L which cannot be matched with this arrangement depend on X_L, in that points of the Smith chart anywhere inside the circle whose diameter is the pair of points given (perhaps inappropriately, as the Smith chart is really a plot of voltage reflection factor) in impedance terms by $z = 0 + j0$ and $z = 0.5 + j0$, cannot be matched.

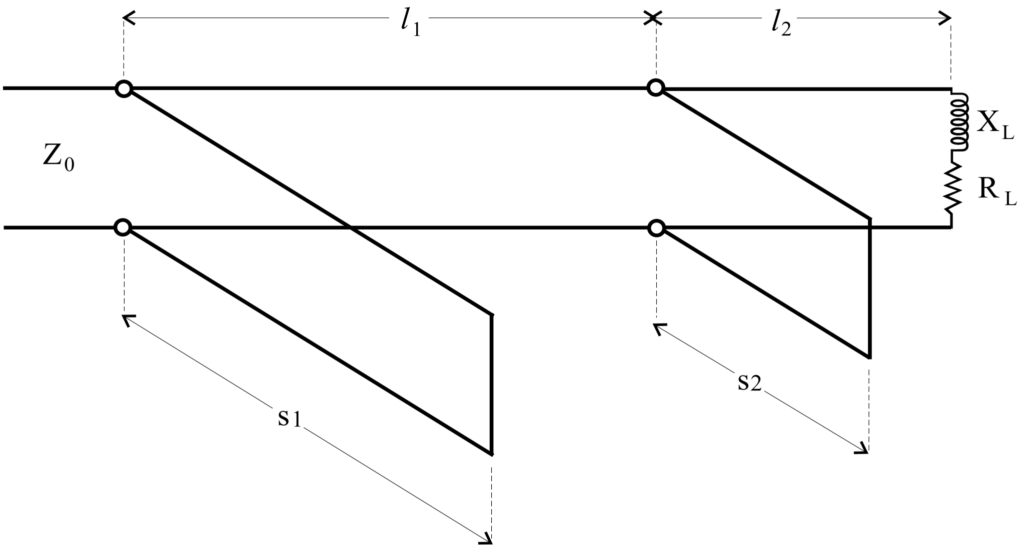


Figure L.16: A double stub matching system.

(b) The values of R_L which cannot be matched with this arrangement again depend on X_L, in that points of the Smith chart anywhere inside the circle whose diameter is the pair of points given (perhaps inappropriately, as the Smith chart is really a plot of voltage reflection factor) in impedance terms by $z = 0 + j0$ and $z = 1.0 + j0$, cannot be matched.

(c) We use an admittance chart because the stubs are in shunt. First we calculate $z_L = 0.5+j1.0$ and then find that $y_L = 0.4-j0.8$. When transformed along the line by $\lambda/10$ this becomes $y'_L = 0.24-0.1$ We then find $y'_L+y_{S1} = 0.24-j0.35$; hence y_{S1} then has length 0.211 λ. The point $y'_L + y_{S1}$ when transformed $3\lambda/8$ toward the generator becomes $1.0 - j1.72$. Thus y_{S2} needed for a match is $+j1.72$. The stub length which provides this susceptance has length 0.417 λ.

(d) The VSWR in the line of length $l2$ can be found from y_L to be 4.2.
The VSWR in the line of length $l1$ can be found from $y_L + y_{S1}$ to be 4.6.

L.7 Plane Wave Problem

A linearly polarised uniform plane wave propagating at a frequency of 100 MHz in the z direction carries a power of 10 μWm^{-2}. Calculate the real and complex Poynting vectors, and write expressions as a function of position for the phasors representing the electric and magnetic fields. Hence derive expressions as a function of position and time for the real field vectors.

Answer

We will assume that the wave is polarised along the x axis, and that the amplitude and phase of the electric field at the origin is given by the complex phasor $\mathrm{E_0}$. In this notation;

$$\mathbf{E}(x,y,z) = \begin{bmatrix} \mathrm{E}_0 \\ 0 \\ 0 \end{bmatrix} e^{-j\beta z}$$

and

$$\mathbf{H}(x,y,z) = \begin{bmatrix} 0 \\ \mathrm{E}_0/\eta \\ 0 \end{bmatrix} e^{-j\beta z}$$

If E_0, which is a complex number, can be expressed in polar form as $E_m e^{j\phi}$, then the real fields are given by

$$\mathsf{E}(x,y,z,t) = \begin{bmatrix} E_m \cos(\omega t - \beta z + \phi) \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathsf{H}(x,y,z,t) = \begin{bmatrix} 0 \\ (E_m/\eta)\cos(\omega t - \beta z + \phi) \\ 0 \end{bmatrix}$$

The real and complex Poynting vectors are then

$$\mathsf{N}\,(x,y,z,t) = \mathsf{E} \times \mathsf{H} = \begin{bmatrix} 0 \\ 0 \\ (E_m^2/\eta)\cos^2(\omega t - \beta z + \phi) \end{bmatrix}$$

and

$$\mathbf{S}(x,y,z) = \frac{1}{2}\,\mathbf{E} \times\ \mathbf{H}^* = \begin{bmatrix} 0 \\ 0 \\ \mathrm{E}_0\ \mathrm{E}_0^*/(2\eta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_m^2/(2\eta) \end{bmatrix}$$

Assuming freee space propagation, $\lambda = 3\mathrm{m}$, so

$$\beta = 2\pi/\lambda = 2.094\ \ \mathrm{m}^{-1}$$

We are also given the power density of $10^{-5}\ \mathrm{Wm}^{-2}$, hence the real part of S_z is

$$\Re\{S_z\} = E_m^2/(2\eta) = 10^{-5}$$

Using the free space value of $\eta = 377\,\Omega$,

$$E_m = \sqrt{377 \times 2 \times 10^{-5}} = 0.0868\ \ \mathrm{V/m}$$

Substituting the value of E_m in the above equations gives the numerical values for most of the questions asked in this problem. We do not have sufficient information to determine the phase angle ϕ.

L.8 Transmission Line Interpretation of Wave Reflection

A linearly polarised uniform plane wave is incident as shown in Figure L.17 from the left on the slab of lossless dielectric material of thickness L.

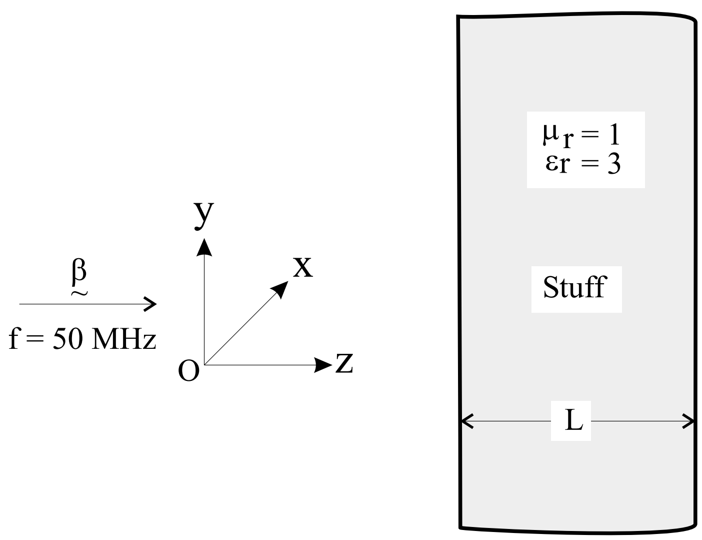


Figure L.17: Transmission through a lossless dielectric slab.

1. Calculate the characteristic impedance and propagation constant for free space and for the medium.

2. Draw a transmission line analogy for the reflection and transmission situation above. Assuming the frequency is 50 MHz and the length L is 0.5 m, use a Smith chart to determine the normalized impedance at the input to the slab, and hence the fraction of the input power reflected.

3. Determine also the fraction of the original incident power which is transmitted to free space on the other side of the slab.

Answers

1. For 50 Mhz the free space wave length is 6 m and the propagation constant is

$$\beta_0 = 2\pi/\lambda = 1.047 \ \mathrm{m}^{-1}$$

For a medium with $\epsilon_r = 3$, the velocity is reduced by a factor $\sqrt{3}$, and the propagation constant is increased by that factor, so the propagation constant in the medium is

$$\beta_m = 1.814 \ \ \mathrm{m}^{-1}$$

The charactersitic impedance of free space is

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega$$

while the charactersitic impedance of the slab is

$$\eta_m = \sqrt{\frac{\mu_0}{\epsilon_r \epsilon_0}} = 217.6\,\Omega$$

2. The transmision line analogy is shown in Figure L.18 in which the line impedances have been normalised with respect to the impedance 217.6 Ω.

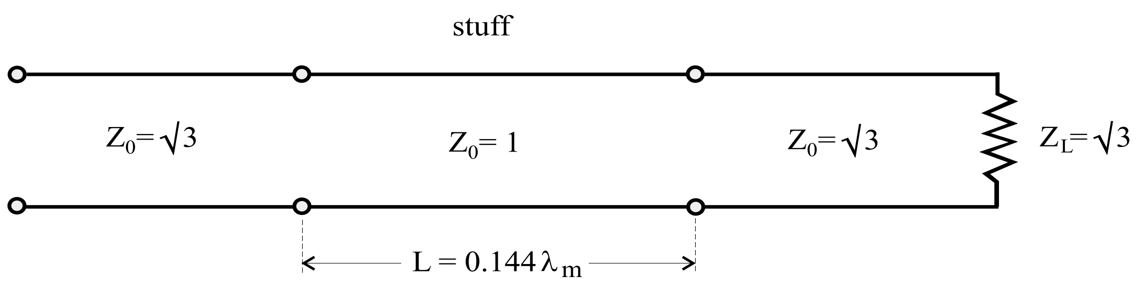


Figure L.18: Transmission line analogy for transmission through a lossless dielectric slab.

the product of line length and propagation constant in the medium is

$$\beta_m L = 0.907$$

More usefully, the line length may be expressed as

$$L = 0.144\ \lambda_m$$

We can then use a Smith chart to calculate from the normalised load impedance $z_L = 1.732 + j0$ the normalised input impedance

$$z_I = 0.78 - j0.43$$

This figure is normalisisd with respect to 217.6 Ω. The un-normalised input impedance is $Z_I = 169 - j93.5\,\Omega$. If we re-normalise this with respect to he characteristic impedance 377 Ω of free space we obtain

$$z_I = 0.448 - j0.248$$

From the Smith chart we obtain for this value of z_I, a value $|\Gamma_v| = 0.411$. Hence

$$\frac{P_r}{P_i} = |\Gamma_v|^2 = 16.9\%$$

3. The remaining 83.1 % of the power is transmitted into the slab, and since the slab is lossless, emerges eventually out of the other side.

L.9 Rectangular Waveguide Problems

1. A standard X-band waveguide, normally used for frequencies in the band 8.2 to 12.4 GHz, has interior dimensions $a = 22.86$ mm and $b = 10.16$ mm. Calculate the cut-off frequency of the dominant, ie. the TE_{10} mode. **Answer**

   The basic relation between complex propagation constant γ and mode numbers is

$$\gamma^2 = \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 - \left(\frac{\omega}{c}\right)^2$$

   where a, b are the dimensions of the wave guide and c is the velocity of light. At the cut-off frequency γ has become zero, so the angular cut-off frequency is given by

$$\left(\frac{\omega_c}{c}\right)^2 = \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

   For mode numbers 1,0 we have then the cut-off frequency

$$f_c = \frac{\omega_c}{2\pi} = \frac{c}{2a}$$

   Inserting the values $c = 3 \times 10^8$ m/s and $a = 22.86$ mm gives the result

$$f_c = 6.56\ \text{GHz}$$

2. Derive an expresson for the cut-off frequencies of the TE_{lm} and TM_{lm} modes, of the form:

$$f_c = B\sqrt{Al^2 + m^2} \tag{L.8}$$

where B is a coefficient which has the units of Hz, and A is a dimensionless coefficient. Hence prepare a *mode chart* showing the cut-off frequencies of the five lowest TE modes.

Answer

We re-arrange the above equation for angular cut-off freqency as

$$\omega_c = \frac{c\pi}{b}\sqrt{\left(\frac{b}{a}\right)^2 l^2 + m^2}$$

Thus

$$f_c = \frac{c}{2b}\sqrt{\left(\frac{10.16}{22.86}\right)^2 l^2 + m^2}$$

The equation has the indicated form with $B = c/(2b) = 14.76$ GHz and $A = 0.1975$. Hence we construct the table

MODE CHART FOR STANDARD RECTANGULAR WAVEGUIDE				
l	m	$Al^2 + m^2$	$\sqrt{Al^2 + m^2}$	f_c
1	0	0.1975	0.4444	6.56 Ghz
2	0	0.7901	0.8888	13.12 Ghz
3	0	1.7775	1.3332	19.68 Ghz
0	1	1.0000	1.0000	14.76 Ghz
1	1	1.1975	1.0943	16.15 Ghz
2	1	1.7901	1.3379	19.75 Ghz
3	2	4.0000	2.0000	29.52 Ghz

Table L.2: Calculations for mode chart of standard rectangular waveguide.

The five lowest modes may be shown in the one-dimensinal chart of Figure L.19.

3. Establish the constraint on the height-to-width ratio of a rectangular waveguide which will maximise the ratio between the cut-off frequencies of the dominant mode and the next highest mode, the latter frequency being in the numerator.

Answer

All the TE andTM modes have an angular cut-off frequency given in terms of mode numbers l and m by

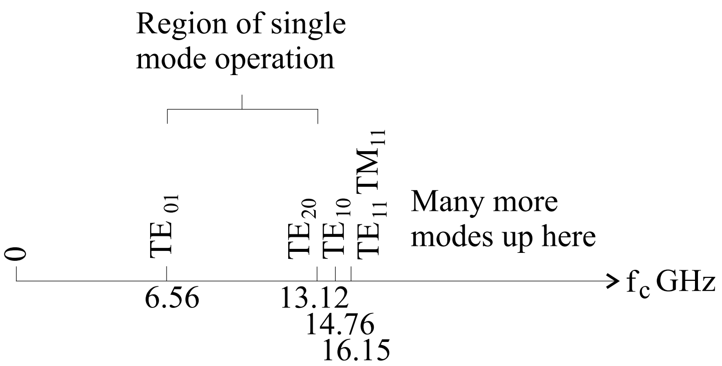


Figure L.19: Mode chart for standard rectangular waveguide.

$$\left(\frac{\omega_c}{c}\right)^2 = \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

Assuming that $a > b$, the dominant mode is the TE_{10}, with angular cut-off frequency $\omega_{c10} = c\pi/a$. The next lowest mode is either the TE_{20} with angular cut-off frequency $\omega_{c20} = 2c\pi/a$, or TE_{01} mode with angular cut-off frequency $\omega_{c01} = c\pi/b$.

We note that for one of these modes, viz the TE_{20} mode, the ratio of its cut-off frequency to that of the dominant mode is 2. To maximise the ratio of the lowest of the non-dominant modes to that of the dominant mode, we must not allow the cut off frequency of the other candidate mode, viz the TE_{01} mode, to fall below twice the cut-off frequency of the dominant mode. That requires that b be at least $0.5a$.

4. Investigate the manner in which:

   (a) the attentuation; and

   (b) the power carrying capacity

   of a waveguide of given width is affected by changes in waveguide height, at a particular operating frequency. Hence derive the height to width ratio which will simultaneously satisfy the constraint established in part (3) above, and which will:

   (c) minimise the attenuation; and

   (d) maximise the power.

   Answer

   Assuming that propagation is in the dominant mode, it should hopefully be clear that the attenuation decreases with increasing waveguide height, while the power carrying capacity increases in proportion to waveguide height (if it is not, please ask the lecturer to explain).

   Thus the waveguide height b which will simultaneously: (i) satisfy the cut-off frequency constraint defined in the previous section; (ii) minimise the attenuation; and (iii) maximise the power carrying capacity, is $b = 0.5a$.

It is therefore not surprising that rectangular waveguide is made approximately to this proportion.

L.10 Conductor Classification

The conductivity of graphite is about 0.12 S/m. Taking its dielectric constant as 5, find an approximate frequency range over which it might be classed as a good conductor.

Answer

For the material to be a good conductor, the conduction current must be much greater than the displacement current, ie $\sigma \gg \omega\epsilon_r\epsilon_0$ or $\omega \ll \sigma/(\epsilon_r\epsilon_0)$. Inserting the parameters for graphite we derive $\omega \ll 2.71 \times 10^9$ rad/s. Hence for a frequency $f \ll 431$ MHz we would consider graphite to be a good conductor.

L.11 Plane Wave Reflection From a Perfect Conductor

A uniform plane wave, polarised as shown in Figure L.20, is incident at an angle θ on the plane boundary $z = 0$ of a perfect conductor. Derive an expression for the real surface current density K, as a function of position and time, which supports this reflection, in terms of the incident electric field amplitude phasor $\mathbf{E}_0$ at the origin.

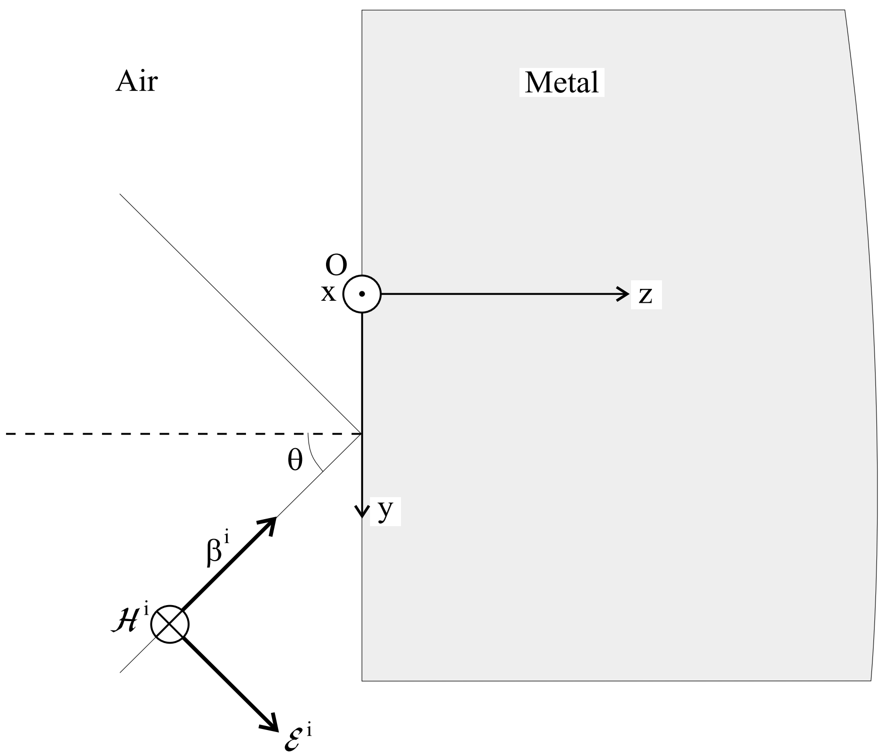


Figure L.20: Plane wave at oblique incidence on a perfect conductor.

Answer

Let us assume that the reflected wave has the angle of reflection equal to the angle of incidence. (We could if pressed produce a good argument based on satisfying the boundary conditions in the plane $z = 0$ that it must be so.) Then the propagation vectors of the incident and reflected waves have the components

$$\boldsymbol{\beta}^i = \begin{bmatrix} 0 \\ -\beta \sin\theta \\ \beta \cos\theta \end{bmatrix}$$

and

$$\boldsymbol{\beta}^r = \begin{bmatrix} 0 \\ -\beta \sin\theta \\ -\beta \cos\theta \end{bmatrix}$$

We will also assume that the reflected wave is plane polarised in the yz plane as is the incident wave. This assumption is derived from the notion that the boundary conditions will require *inter alia* that the total electric field in the x direction is zero, and the incident field has no compont in the x direction. Thus the electric fields of the incident and reflected waves take the form

$$\mathbf{E}^i = \mathrm{E}_0 \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} e^{j\beta(y\sin\theta - z\cos\theta)} \tag{L.9}$$

$$\mathbf{E}^r = \mathrm{E}_1 \begin{bmatrix} 0 \\ -\cos\theta \\ \sin\theta \end{bmatrix} e^{j\beta(y\sin\theta + z\cos\theta)} \tag{L.10}$$

where the amplitude E_1 is to be found.

The boundary conditions we must satisfy are $\mathbf{E}_t = 0$, ie $E_x = 0$ and $E_y = 0$, in the plane $z = 0$. We have already satisfied the condition $E_x = 0$ by the way we have constructed $\mathbf{E}^r$. We can now satisfy $E_y = 0$ by setting

$$\mathrm{E}_1 = \mathrm{E}_0$$

The magnetic fields associated with the two waves are then

$$\mathbf{H}^i = \frac{\mathrm{E}_0}{\eta} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^{j\beta(y\sin\theta - z\cos\theta)} \tag{L.11}$$

$$\mathbf{H}^r = \frac{\mathrm{E}_0}{\eta} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^{j\beta(y\sin\theta + z\cos\theta)} \tag{L.12}$$

These fields have been obtained by using the usual conditions for a TEM wave, viz that $\mathbf{E}$, $\mathbf{H}$ and $\vec{\beta}$ form a right had set of mutually orthogonal vectors, that the magnitude ratio is η, and the electric and magnetic fields are in time phase.

The total magnetic field in the plane $z = 0$ is therefore

$$\mathbf{H}(x, y) = \frac{-2\mathrm{E}_0}{\eta} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{j\beta y \sin\theta} \tag{L.13}$$

The accompanying surface current phasor $\mathbf{K}$ is directed along the y axis and has the amplitude

$$\mathrm{K}_y = \frac{2\mathrm{E}_0}{\eta} e^{jy \sin\theta} \tag{L.14}$$

From this phasor we construct by the usual rules the real surface current

$$K_y = \frac{2E_0}{\eta} \cos(\omega t + \beta y \sin\theta) \tag{L.15}$$

This is a wave of surface current directed in the y direction and travelling in the $-y$ direction, ie upward.

In transforming to the real variables we have made the tacit assumption that the phasor E_0 representing the value at the origin of the electric field of the incident wave is real. If instead, this phasor had a phase angle ϕ, we would have to add ϕ to the argument of the cosine function in the last expression.

L.12 Boundary Conditions

1. Defining carefully the notation introduced, state in integral form and in the time domain Ampere's law as modified by Maxwell. Interpret physically all terms of the equation.

2. Using the above law, establish the boundary conditions which obtain for the tangential components of the magnetic field adjacent to but on opposite sides of the plane boundary between an insulating and a perfectly conducting medium.

Answer

We are yet provide a solution to this problem. Since it is entirely lecture note material, maybe we will not. But who knows? Watch this space!

L.13 Plane Waves

The electric field intensity associated with a plane electromagnetic wave incident in vacuum on the boundary $z = 0$ of a perfect conductor is given in SI units by the expression:

$$\mathsf{E}(x, y, z, t) = 10 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos(2\pi f t - z) \;\; \mathrm{Vm}^{-1}. \tag{L.16}$$

Determine:

1. An expression as a function of position and time for the magnetic field of the incident wave.
2. A numerical value for the frequency f of the wave.
3. A numerical value for the power density carried by the wave.
4. An expression as a function of position and time for the real surface current density set up by the wave at its reflection in the plane $z = 0$.

Anwer

It is possible that this problem will bring back old and pleasant memories.

1. In free space for a propagating wave the electric and magnetic fields are in time phase, are mutually orthogonal and are orthogonal to the propagation direction, with E, H and $\vec{\beta}$ (in that order) forming a right hand system.

   Since in this case the wave is propagating in the $+z$ direction and the electric field is in the $+y$ direction we expect the magnetic field to be in the $-x$ direction to make a right hand system. The peak value of the magnetic field will be $(10/\eta)\, Am^{-1}$ where $\eta = 377\,\Omega$. Thus

$$\mathsf{H}(x, y, z, t) = \frac{10}{377}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cos[2\pi(ft - z)] \ \ \mathrm{Am}^{-1} \tag{L.17}$$

   We will obtain a particular value for f in the next section.

2. The general form of the position and time dependence for a sinusoidal travelling wave with propagation constant β in the z direction is $\cos(\omega t - \beta z)$. Comparing this form with that given, we see that $\beta = 2\pi \ \ \mathrm{m}^{-1}$. The wave length λ is thus one metre.

   The relation between frequency, wave length and velocity being $v = f\lambda$, and the velocity of electromagnetic waves in free space being $3 \times 10^8 \ \ \mathrm{ms}^{-1}$, we conclude the frequency f is 300 MHz.

3. The electric and magnetic fields of the wave can be represented by complex phasors (which represent peak not rms values) $\mathrm{E}_y = 10e^{-j\beta z}$ V/m and $\mathrm{H}_x = (-10/377)e^{-j\beta z}$ A/m. From these phasors we see the complex Poynting vector has only a z component

$$S_z = -\frac{1}{2}\ \mathrm{E}_y\ \mathrm{H}_x^* \tag{L.18}$$

   ie

$$S_z = \frac{50}{377}\ \ \mathrm{Wm}^{-2} \tag{L.19}$$

   This number is purely real and gives the power per unit area carried by the wave in the $+z$ direction.

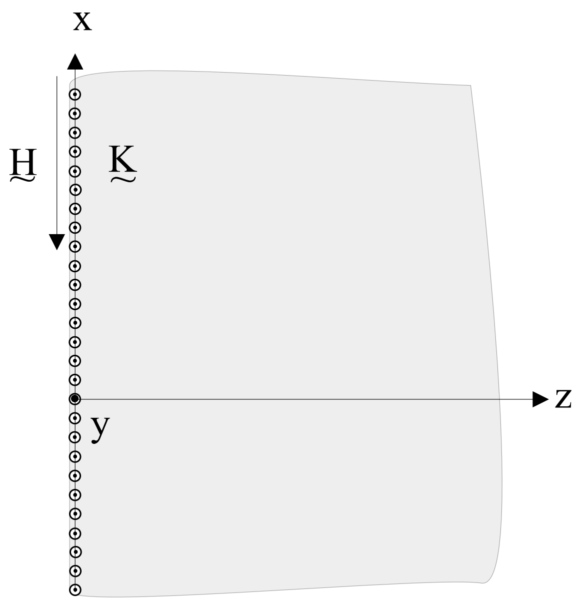


Figure L.21: Surface current at a metal boundary

4. When such a wave encounters at right angles a perfectly conducting plane, an equal-amplitude oppositely-directed reflected wave is set up. The phase relations between the fields of the incident wave and those of the reflected wave are such that the electric fields exactly *cancel* and the magnetic fields *double*.

   Thus in the plane $z = 0$ we expect to have a magnetic field

$$\mathsf{H}(x, y, 0, t) = \frac{20}{377} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cos(2\pi f t)\ Am^{-1} \tag{L.20}$$

   When such a magnetic field exists adjacent to the boundary of a perfect conductor there is required a surface current density equal in magnitude and in time phase with the field, but in the orthogonal direction, with the sense determined by the right hand rule, as shown in Figure L.21.

   Thus a surface current density in the $+y$ direction will produce on the $z < 0$ side of the xy plane a magnetic field in the $-x$ direction. Thus the expression for the surface current density is

$$\mathsf{K}(x, y, 0, t) = \frac{20}{377} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos(2\pi f t)\ \ \mathrm{Am}^{-1} \tag{L.21}$$

L.14 Radiation

1. (a) Calculate the radiation resistance of a short dipole of length 10 cm, carrying an oscillating current of 100 mA peak at 150 MHz, the current being assumed

to be uniform along its length.

(b) Calculate the total power radiated by the dipole.

2. If the dipole is located at the origin and is directed along the z axis, calculate the strength of the electric field radiated at a distance of 1000 m:

   (a) along the z axis; and

   (b) along the x axis.

3. Calculate the skin depth in copper (conductivity 5.8×10^7 S/m) at 150 MHz, and hence derive the series loss resistance of the above dipole assuming it has the form of a copper rod 2 mm in diameter. Compare this value with the radiation resistance.

4. It may be argued that the assumption of a uniform current distribution on the antenna is unrealistic. What do you consider would be a more reasonable assumption to make for the distribution of current, and how would the calculations of radiation resistance and loss resistance then be modified?

5. The calculations have necessarily ignored the question of the input reactance of the dipole. What form of input reactance (high, low, zero, capacitive, inductive) would you expect?

6. To what extent would matching of the dipole input impedance to the output impedance of commonly available generators affect the overall efficiency of the system as a radiator of electromagnetic energy?

Answer

1. (a) For a short dipole of length L carrying a uniform current the radiation resistance is

$$R_r = \frac{\eta}{6\pi}(\beta L)^2 = \left(\frac{2\pi}{3}\right)\eta\left(\frac{L}{\lambda}\right)^2 \approx 20(\beta L)^2 \tag{L.22}$$

   At 150 Mhz with $L = 100$mm, we have $R_r = 1.97\,\Omega$

   (b) The total power radiated is $W = \frac{1}{2}R_r|\,\mathrm{I}|^2 = 9.85$ mW.

2. (a) The dipole does not radiate in the direction of its axis, so the electric field at at large distance in the z direction is zero.

   (b) Since the z axis is the axis of the dipole the x axis is the direction of greatest radiated power density. We can then make use of the known gain of the dipole to calculate from the expressionfor power flow per unit area

$$\frac{|\,\mathrm{E}|^2}{2\eta} = g\frac{W}{4\pi r^2} \tag{L.23}$$

   Substituting the values $\eta = 120\pi$, $g = 1.5$, $W = 9.85$mW, and $r = 1000$m gives

$$|\,\mathrm{E}| = 0.942 \ \ \mathrm{mV/m} \tag{L.24}$$

As an alternative method of calculation we may make use directly of the result derived in lectures for the electric field of a small dipole , viz

$$\mathrm{E}_\theta = \frac{j\omega\mu_0 \, \mathrm{I}L \sin\theta e^{-j\beta r}}{4\pi r} \tag{L.25}$$

In the eqatorial plane, $\sin\theta = 1$. Substituting the values $\omega = 300\pi \times 10^6$ rad/s, $\mu_0 = 4\pi \times 10^{-7}$ H/m, I $= 100$ mA, $L = 0.1$ m and $r = 1000$ m gives the field

$$|\ \mathrm{E}_\theta| = 0.942 \ \ \mathrm{mV/m} \tag{L.26}$$

3. The skin depth at angular frequency ω is given by

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \tag{L.27}$$

When $\omega = 300\pi \times 10^6$ rad/s, $\mu_0 = 4\pi \times 10^{-7}$ H/m, and $\sigma = 5.8$ S/m, we find that

$$\delta = 5.4\,\mu \text{ m} \tag{L.28}$$

Thus the surface resistivity is

$$R_s = \frac{1}{\delta\sigma} = 3.20 \text{ m}\Omega \text{ per square} \tag{L.29}$$

.

The total series loss resistance of the rod of radius a and length L is

$$R_l = \frac{LR_s}{2\pi a} = \frac{0.1 \times 3.2 \times 10^{-3}}{2\pi \times 10^{-3}} = 50.0 \ \ \mathrm{m}\Omega \tag{L.30}$$

This value is significantly less than the radiation resistance, and appears to suggest that the dipole could be an efficient radiator. The matters discussed below, however, reveal that significant practical problems related to matching the entire dipole impedance to a signal generator may prevent the realisation of an efficient radiating system with a dipole of this length.

4. If we assume that the dipole is centre fed we may be tempted to consider, instead of the uniform current distribution assumed in the short dipole analysis, a non-uniform current distribution, maximum at the centre and tapering linearly to zero at the ends. Such a distribution would not, however, produce the crowding of charge toward the ends of the a thin rod whch is characteristic of the solution of an electrostatic field problem in the vicinity of a pointed conductor.

   Probably the true distribution is somewhere between the uniform and linearly tapering versions just discussed. It is hopefly clear that if this be so, the radiation resistance calculated on the uniform current assumption overestimates the true value, but by a factor of less than four.

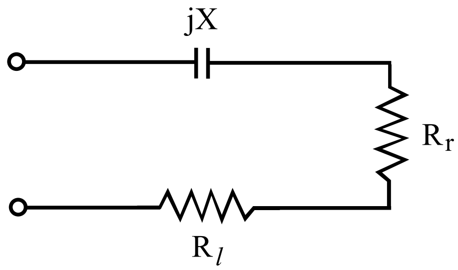


Figure L.22: Equivalent circuit of a short electric dipole.

5. One should expect the input reactance to be large and negative, ie corresponding to a small capacitance. The equivalent circuit of the dipole is shown in Figure L.22.

   In fact detailed analysis using techniques of numerical analysis not presented n this course shows that the reactance is $-j2600\,\Omega$.

6. To match the dipole to the generator, this reactance would need to be tuned. At the frequency of 150 MHz, such a coil would require an inductance of $2.8\,\mu$ H to provide the required reactance of $j2600\,\Omega$. To minimise losses a physically large coil would be needed, probably larger than the dipole, but it is doubtful that even then the coil losses could be made small in relation to the dipole radiation resistance determined above. Moreover, a large coil would have a significant self capacitance, and inconsequence too low a self resonant frequency, and would not act as an inductor at the operating frequency. So efficient matching is probably not possible.

L.15 Transmission Line Losses

1. Calculate the attenuation constant α in nepers per metre, and the transmission loss in dB per metre, for an air-filled co-axial line of inner and outer conductor radii a and b, respectively, in terms of conductor material conductivity σ and the proportions of the line.

2. How does this result scale with frequency?

3. What are the losses for an air-filled copper co-axial line of inner conductor radius 2 mm and outer conductor radius 7 mm at frequencies of:

   (a) 100 kHz; and

   (b) 10 Mhz?

4. What length of this cable would you use as a high power dummy load to achieve, at a frequency of 1.0 GHz, and for any value of load impedance, an input VSWR of less than or equal to 1.5?

Answer

1. For the co-axial line in which the phasor representing the circumferential component of magnetic field at the inner conductor is H_0, we have derived the results

$$W_L = \pi R_s |\,\mathrm{H}_0|^2 a(a+b)/b \tag{L.31}$$

$$W_T = \pi\eta |\,\mathrm{H}_0|^2 a^2 \log_e(b/a) \tag{L.32}$$

where W_L is the power lost in the walls per unit length of the line, W_T is the power transmitted along the line, and R_s is the surface resistivity due to skin effect. If σ is the material conductivity and δ is the skin depth, the R_s is given by

$$R_s = \frac{1}{\delta\sigma} = \sqrt{\frac{\omega\mu_0}{2\sigma}} \tag{L.33}$$

From conservation of energy applied to the length δz of coaxial line shown in Figure L.23 we conclude that

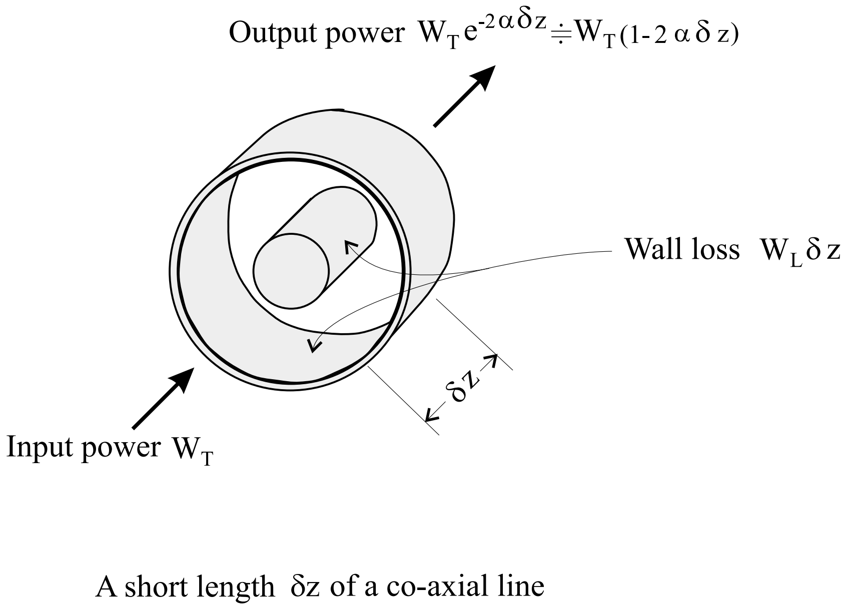


Figure L.23: Coaxial line section with wall loss.

$$\alpha = \frac{W_L}{2W_T} \tag{L.34}$$

Hence the desired expression for α is

$$\begin{aligned}\alpha &= \sqrt{\frac{\omega\mu_0}{2\sigma}}\frac{1}{\eta}\frac{a+b}{2ab\log_e(b/a)}\\ &= \sqrt{\frac{\omega\epsilon}{2\sigma}}\frac{a+b}{2ab\log_e(b/a)}\end{aligned}$$

This is the attenuation constant in nepers per metre. Now for one neper attenuation, we have an amplitude reduction factor of e, and a power reduction factor of e^2, which comes in more usual terms to 8.686 dB. Hence the loss of the line in dB per metre is

$$8.686\sqrt{\frac{\omega\epsilon}{2\sigma}}\frac{a+b}{2ab\log_e(b/a)} \;\; \text{dB/m} \tag{L.35}$$

2. Clearly the loss scales as $\sqrt{\omega}$.

3. (a) For copper at 100 kHz and the indicated proportions of the line, we have

$$\begin{aligned} \omega &= 2\pi \times 10^5 \;\; \text{rad/s} \\ \epsilon &= 8.854 \times 10^{-12} \;\text{F/m} \\ \sigma &= 5.8 \times 10^7 \;\text{S/m} \\ a &= 2 \;\text{mm} \\ b &= 7 \;\text{mm} \end{aligned}$$

   Hence $\alpha = 5.62 \times 10^{-5}$ neper/m i.e. 4.88×10^{-4} dB/m

   (b) At 10 Mhz, we will have an attenuation ten times this amount, ie 4.88×10^{-3} dB/m.

4. To achieve a VSWR of ≤ 1.5 we must have a reflection coefficient $|\Gamma_v| \leq 0.2$. The power refelction coefficient is thus ≤ 0.04, i.e. the reflection loss for power sent down the line is 14 dB. The worst case corresponds to a lossless termination, in which case we need at least 7 dB of attenuation for each of the two passages (forward and reverse) of the signal along the line. Since at 1 Ghz, the loss is expected to be 0.0488 dB/m, we need $7/0.0488 = 143$ metres of cable.

L.16 More on Coaxial Line Losses

1. How do the losses of a co-axial line vary with Z_0, if the outer diameter is kept fixed?

2. Is there an optimum Z_0, and if so what is its approximate value, if the dielectric constant is 2.25?

Answer

1. We use the results for Z_0 and α

$$\begin{aligned} Z_0 &= \sqrt{\frac{\mu_0}{\epsilon_r\epsilon_0}}\frac{\log_e(b/a)}{2\pi} \\ \alpha &= \sqrt{\frac{\omega\epsilon_r\epsilon_0}{2\sigma}}\frac{(a+b)}{2ab\log_e(b/a)} \end{aligned}$$

The second result can be re-arranged as

$$\alpha = \sqrt{\frac{\omega\epsilon_r\epsilon_0}{2\sigma}}\frac{1+(b/a)}{2b\log_e(b/a)} \tag{L.36}$$

As, for a given value of b, the value of a decreases from its maximum value of b down to zero, the impedance Z_0 increases from 0 to ∞. At the same time, the above function for α decreases from ∞ at a value of $b/a = 1$, down to a minimim value which occurs at about $(b/a) = 3.6$, and then increases again to ∞ as $a \to 0$.

2. The minimum value of Z_0 is given by

$$Z_0 = \frac{\eta}{2\pi}\frac{log_e(3.6)}{\sqrt{\epsilon_r}} \tag{L.37}$$

For a value of $\epsilon_r = 2.25$, the characteristic impedance of the minimum loss cable is 51 Ω. We note that rf cables are very commonly of 50 Ω characteristic impedance.

L.17 Skin Depth and Waveguide Loss

1. Calculate the skin depth and surface resistivity in copper for frequencies of:
   (a) 1 MHz;
   (b) 100 Mhz; and
   (c) 1 GHz.
2. Derive an expression for the attenuation constant for the dominant, i.e. TE_{10}, mode of a rectangular wave guide of height b and width a.
3. Calculate the attenuation, in dB/m, for copper waveguide at a frequency of 10 GHz.

Answer

1. (a) Using the formulae

$$\begin{aligned}\delta &= \sqrt{\frac{2}{\omega\mu_0\sigma}}\\ R_s &= \sqrt{\frac{\omega\mu_0}{2\sigma}}\end{aligned}$$

with $\omega = 2\pi \times 10^6$, $\mu_0 = 4\pi \times 10^{-7}$, and $\sigma = 5.8 \times 10^7$, all in SI units, we find at a frequency of 1 MHz, $\delta = 66.1\ \mu$m and $R_s = 260.9\ \mu\Omega$ per square.

   (b) At this frequency, being 100 times the above, the skin depth will be reduced by a factor of 10 and the surface resistivity will be increased by a factor of 10 from the values in the previous section.

(c) At this frequency, being 10 times the above, the skin depth will be reduced by a factor of $\sqrt{10}$ and the surface resistivity will be increased by a factor of $\sqrt{10}$ from the values in the previous section.

2. We use the results

$$\begin{aligned} \alpha &= \frac{W_L}{2W_T} \\ W_L &= \int_{\text{Periphery}} \frac{1}{2} R_s \, \mathbf{H} \cdot \, \mathbf{H}^* dr \\ W_T &= \int_{\text{Cross section}} \Re \left\{ \frac{1}{2} \, \mathbf{E} \times \, \mathbf{H}^* \right\}_z dxdy \end{aligned}$$

Using the dominant mode fields supplied and performing the integrations around the waveguide periphery and over the waveguide cross section we find that

$$\begin{aligned} W_L &= \frac{1}{2} R_s H H^* (2b + a\beta_0^2/\beta_c^2) \\ W_T &= \eta (ab/4)(\beta\beta_0/\beta_c^2) H H^* \end{aligned}$$

We also use the result for R_s

$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} \tag{L.38}$$

to assemble the final result

$$\alpha = \sqrt{\frac{\omega\mu_0}{2\sigma}} \frac{2b\beta_c^2 + a\beta_0^2}{\eta ab\beta\beta_0} \tag{L.39}$$

3. Using the values $a = 22.86$ mm, $b = 10.16$ mm, $\omega = 2\pi \times 10^{10}$ rad/s, $\mu_0 = 4\pi \times 10^7$ H/m, $\sigma = 5.8 \times 10^7$ S/m, $\eta = 120\pi$ Ω, $\beta_c = \pi/a$, $\beta_0 = \omega/c$, we obtain the result

$$\alpha = 14.53 \times 10^{-3} \text{ neper/m} \tag{L.40}$$

Multiplication by the factor of $20 \log_{10} e = 8.686$ gives the attenuation of 0.125 dB/m.

L.18 Double Stub Tuner

A load of impedance $(11 - j4)$ Ω is connected via a length of 0.34 wave lengths of transmission line of characteristic impedance 50 Ω to a double stub tuner, as shown in the Figure L.24.

The tuner consists of a length of 0.3 wave lengths of transmission line of characteristic impedance 70 Ω at each end of which is connected a variable length short circuited stub transmission line of characteristic impedance 100 Ω.

The tuner is intended to produce a match of the load to the long 50 Ω transmission line shown at the left of the figure. Determine stub lengths which will accomplish this result.

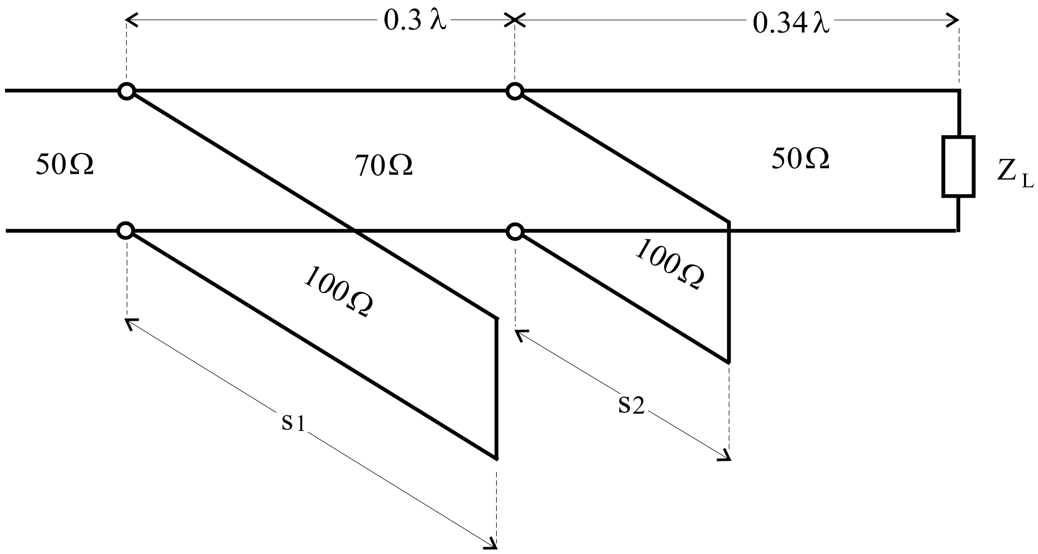


Figure L.24: A double stub tuner.

Answer

The Smith chart solution to this problem is displayed in Figure L.25. The solution procedure is

- Normalise the load impedance with respect to 50 Ω and enter the Smith chart at z_L so found.
- Reflect z_L in the origin to find y_L.
- Transform y_L by 0.34 λ toward the generator to find $y'_L(50)$.
- Un-normalise $y'_L(50)$ with respect to 50 Ω to find Y'_L and re-normalise that with respect to 70 Ω to find $y'_L(70)$.
- Construct the constant conductance circle through $y'_L(70)$.
- Rotate the $g = 70/50 = 1.4$ circle of the chart an amount 0.3λ toward the load to obtain a dotted locus shown in Figure L.25.
- Look for points X and X′ of intersection of the rotated $g = 1.4$ circle with the constant conductance circle through $y'_L(70)$. We will proceed with just the solution based on the point X.
- Rotate the point X toward the generator by 0.3 λ to find the point Y. This point shoud lie on the $g = 1.4$ locus.
- Read the $b_Y = -1.08$ at the point Y on the chart.

- Calculate the value $b_{S1}(70) = -b_Y$ to be provided by stub S1.
- Un-normalise that value just obtained with respect to 70 Ω and re-rormalise with respect to 100 Ω to find the vlaue $b_{S1}(100)$ to be provided by the 100 Ω stub S1.
- Enter the chart on the periphery at the point $b_{S1}(100)$ so obtained and rotate around the periphery in the direction of the load to the $y \to \infty$ point representing the short circited end of the stub to find the stub lengh in terms of λ.
- Return to the point y'_L on the chart and find the value of susceptance b'_L at that point.
- Read also the value b_X of suseptance at the point X.
- The difference is the normalised susceptance provided by the stub S2, ie $b_{S2}(70) = b_X - b'_L$. Calculate this value as -0.592.
- Un-normalise that value just obtained with respect to 70 Ω and re-rormalise with respect to 100 Ω to find the value $b_{S2}(100)$ to be provided by the 100 Ω stub S2.
- Enter the chart on the periphery at the point $b_{S2}(100)$ so obtained and rotate around the periphery in the direction of the load to the $y \to \infty$ pont representing the short circited end of the stub to find the stub lengh in terms of λ.

The results are the length of S1 is 0.408 λ and the length of S2 is 0.139 λ. There is of course another solution based upon the point X′, but we have not determined the details.

L.19 Transients on Transmisison Lines

In the circuit diagram below, the circuit S acts as a source of short duration transient signals, the transmissions system T contain two dissimilar cables, and the wideband oscilloscope C has an input impedance of 50 Ω, unaccompanied by any capacitance. The capacitor is initially charged to a voltage of 10V.

1. Neglecting the loading of the transmission lines, calculate the transient voltage developed across the 1 Ω resistor after the switch is closed.
2. Calculate the reflection and transmission factors where the 70 Ω line meets the 50 Ω line and where the 50 Ω load terminates the 70 Ω line.
3. Calculate the propagation times along the 4 m and 2 m lengths of line.
4. Sketch the form of voltage detected by the oscilloscope for the first 50 ns after the switch is closed.

Answer

1. The transient source voltage is $v(t) = 10e^{-10^9 t}$ V.

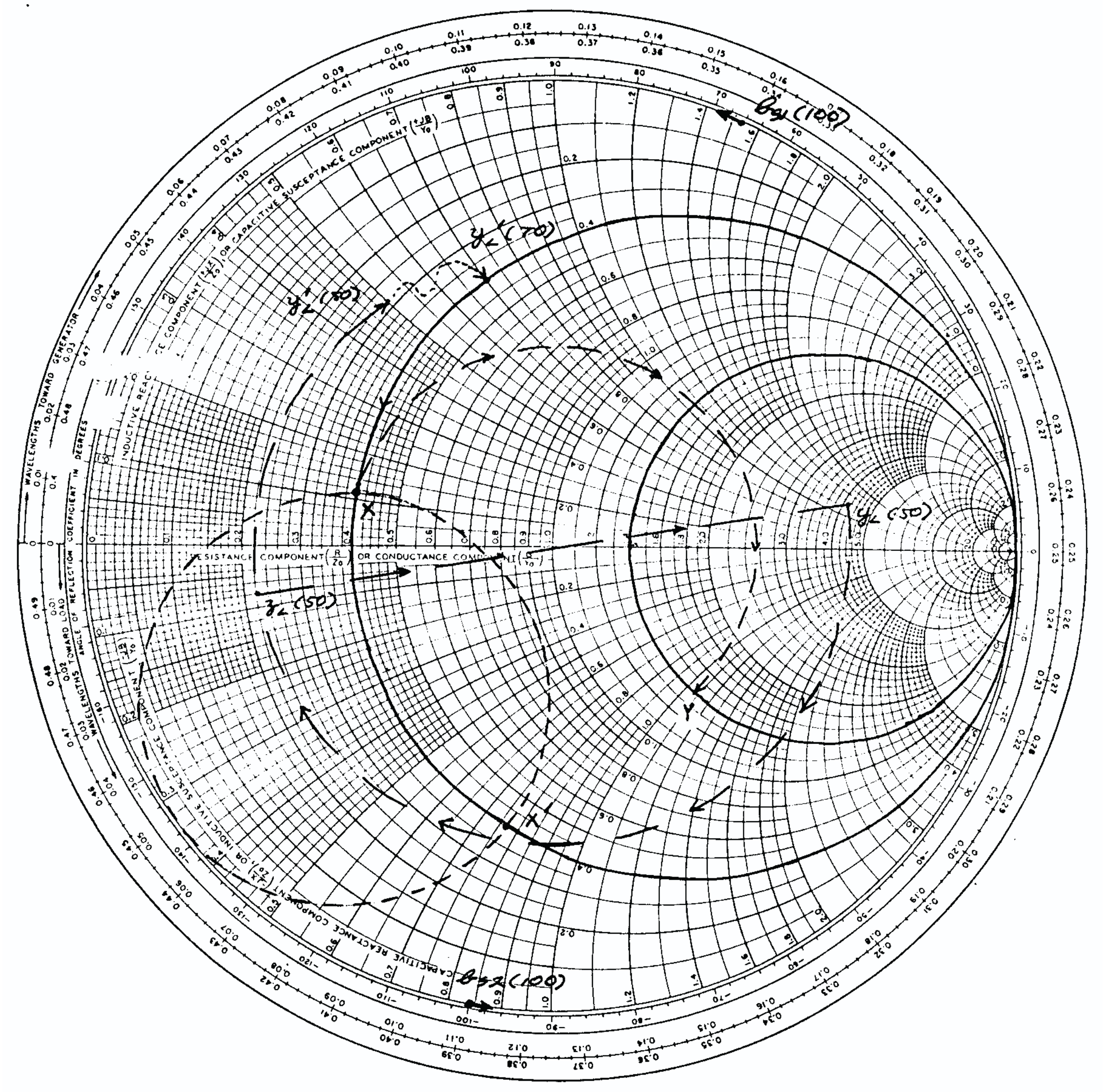


Figure L.25: Smith chart solution for double stub tuner problem.

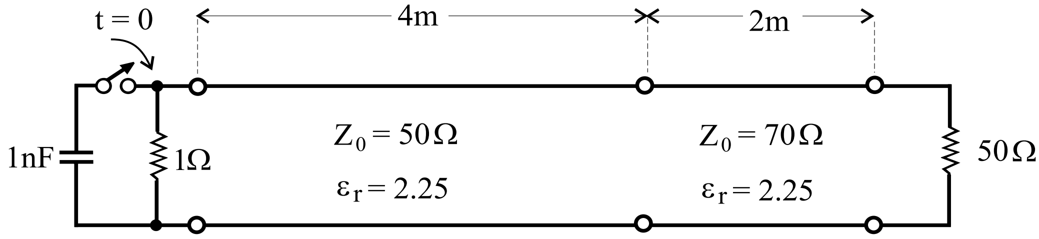


Figure L.26: Transmission lines feeding an oscilloscope

2. The reflection factor looking into the 70 Ω line from the 50 Ω line is $\Gamma_v = \frac{70-50}{70+50} = \frac{1}{6}$.

   The reflection factor looking into the 50 Ω load from the 70 Ω line is $\Gamma_v = \frac{50-70}{50+70} = -\frac{1}{6}$.

3. The propagation velocity on both lines is 2×10^8 m/s. The propagation time along the 4m line is 20 ns and along the 2m line is 10 ns.

4. Although there can be many reflections up and down the several lines, for the first 50 ns, only the direct wave which is launched from the source into the first line, is then partially transmitted into the second line, and then propagates down the second line to the load, where it combines with its reflected wave at that point to from the load voltage, will be seen.

   The initially launched wave has the same form as the source voltage given above. At the first interface, the voltage reflection factor is $\frac{1}{6}$, and thus the transmission factor to the second line is $\frac{7}{6}$. At the second interface (i.e with the load) the voltage reflection factor is - $\frac{1}{6}$ so the voltage transmission factor to the load is $\frac{5}{6}$. The product of the transmison factors is thus $\frac{35}{36}$. The voltage reaching the load for the first 50 ns is thus

$$v_L(t) = \frac{35}{36} e^{-10^9 t} \text{ V} \tag{L.41}$$

L.20 An Old Friend

The parallel plate capacitor shown below is partly filled with a dielectric slab of dielectric constant 2.25, the remaining space being occupied by air. A voltage of 10 V is established between the plates.

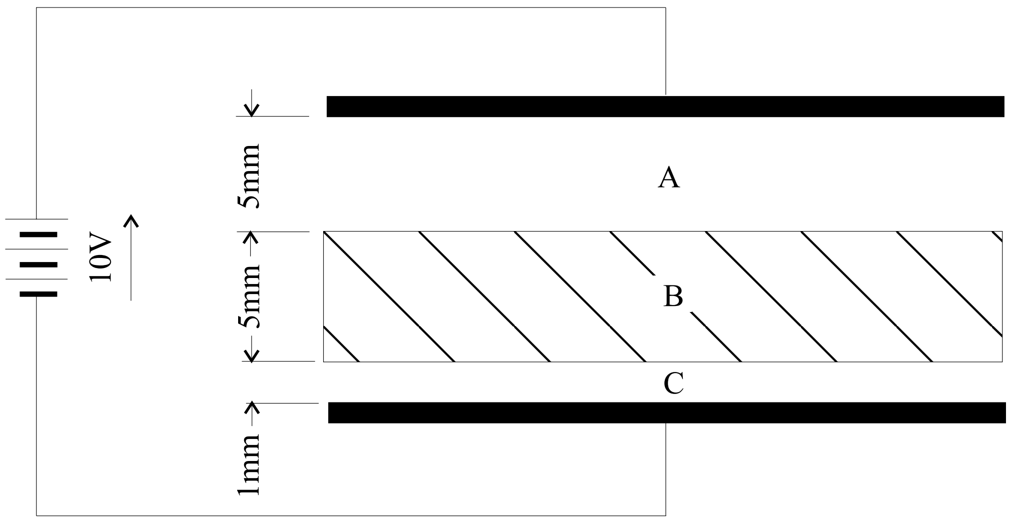


Figure L.27: Partially filled parallel plate capacitor.

Calculate in an appropriate order:

1. The electric field in the regions A, B and C.
2. The electric flux density in those regions.
3. The surface charge density on each of the plates.
4. The induced surface charge on each of the dielectric surfaces.

Answer

The appropriate starting point is to recognise that the electric flux density D which originates on and is normal to the upper plate and is equal to the surface charge density thereon remains the same in regions A, B and C, and is equal to minus the surface charge density on the lower plate. Thus

$$D_A = D_B = D_C \tag{L.42}$$

Since D and E are in a linear medium simply proprtional, this result can be expressed as

$$E_A = \epsilon_r E_B = E_C \tag{L.43}$$

Now the total potential difference between the plates is given by

$$E_A t_A + E_B t_B + E_C t_C = 10 \text{ V} \tag{L.44}$$

Substituting for E_B and E_C in terms of E_B from above gives

$$E_A(t_A + t_B/\epsilon_r + t_C) = 10 \text{ V} \tag{L.45}$$

1. Making use of the given values of t_A, t_B, t_C, and ϵ_r we obtain

$$\begin{aligned} E_A &= 1216.2 \ \text{Vm}^{-1} \\ E_B &= 540.5 \ \text{Vm}^{-1} \\ E_C &= 1216.2 \ \text{Vm}^{-1} \end{aligned}$$

2. Thus

$$D_A = D_B = D_C = 10.78 \ \text{nCm}^{-2} \tag{L.46}$$

3. The surface charge density at the top plate is 10.78 nCm^{-2} and at the bottom plate is -10.78 nCm^{-2}.

4. Now the polarisation can be found from

$$P_B = \epsilon_0(\epsilon_r - 1)E_B = 5.892 \ \text{nCm}^{-2} \tag{L.47}$$

The induced surface charge density at the top of the dielectric is therefore 5.892 nCm^{-2} and at the bottom is -5.892 nCm^{-2}.

L.21 More Exercises on Notation and Plane Waves

1. Construct a vector phasor to represent the magnetic field which is expressed in SI units by the expression:

$$\mathsf{H}(x, y, z, t) = \begin{bmatrix} 0 \\ e^{-4z} \cos(1 + 1.2 \times 10^9 t - 5y) \\ 0 \end{bmatrix} \tag{L.48}$$

2. Determine for this wave:

   (a) the frequency in Hz;

   (b) the complex propagation vector;

   (c) the planes of constant phase; and

   (d) the planes of constant amplitude.

3. For the above wave use the Maxwell equation giving the curl of the magnetic field to determine the vector phasor representing the corresponding electric field, assuming that the fields exist in empty space.

   For interest it may be stated that these fields are those which may be found in empty space adjacent to a plane boundary with a dielectric region in which is occurring total internal reflection of a uniform plane wave incident at a suitable angle on the boundary from within the dielectric.

4. Does the field described by the given equations itself correspond to a uniform plane wave? If so what is the direction of that wave? If not, why not?

Answer

1. The factor $\cos(\omega t - \beta y + \phi)$ in the time domain will have in the frequency domain representation a factor $e^{-j}e^{-j\beta y}$, where $\beta = 5$. Thus

$$\mathbf{H}(x, y, z) = \mathrm{H}_0 \begin{bmatrix} 0 \\ e^{j-5jy-4z} \\ 0 \end{bmatrix} = \mathbf{H}_0 e^{-\vec{\gamma}\cdot\ \mathbf{r}} \tag{L.49}$$

   where $\vec{\gamma} = \vec{\alpha} + \vec{\beta}$ are given below.

$$\vec{\gamma} = \begin{bmatrix} 0 \\ 5j \\ 4 \end{bmatrix}; \ \vec{\alpha} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}; \ \vec{\beta} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \tag{L.50}$$

2. (a) The angular frequency is $\omega = 1.2 \times 10^9$ rad/s. Thus the frequency is $\frac{1.2\times10^9}{2\pi} =$ 191 MHz.

(b)

$$\vec{\gamma} = \begin{bmatrix} 0 \\ 5j \\ 4 \end{bmatrix}; \ \vec{\alpha} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}; \ \vec{\beta} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \tag{L.51}$$

(c) Planes of constant phase are those perpendicular to the y axis.

(d) Planes of constant amplitude are those perpendicular to the z axis.

3. The Maxwell curl equation for magnetic field is

$$\mathbf{curl}\ \mathbf{H} = \ \mathbf{J} + j\omega\ \mathbf{D} = \ \mathbf{J} + j\omega\epsilon\ \mathbf{E} \tag{L.52}$$

Since $\mathbf{J} = 0$ and $\mathbf{D} = \epsilon\ \mathbf{E}$ we have

$$\mathbf{E} = \frac{1}{j\omega\epsilon} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{H}_x & \mathrm{H}_y & \mathrm{H}_z \end{vmatrix} \tag{L.53}$$

in which we may replace $\frac{\partial}{\partial x}$ by 0, $\frac{\partial}{\partial y}$ by $-5j$ and $\frac{\partial}{\partial z}$ by -4. Substituting for both the derivatives and for the components of $\mathbf{H}$, we have

$$\mathbf{E} = \frac{1}{j\omega\epsilon_0} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -5j & -4 \\ 0 & 1 & 0 \end{vmatrix} e^{-\vec{\gamma}\cdot\ \mathbf{r}} \tag{L.54}$$

Expanding gives

$$\begin{bmatrix} \mathrm{E}_x \\ \mathrm{E}_y \\ \mathrm{E}_z \end{bmatrix} = \frac{1}{j\omega\epsilon_0} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} e^{-\vec{\gamma}\cdot\ \mathbf{r}} \tag{L.55}$$

We note that the complex Poynting vector, if it were calculated, would have in the z direction a real part of zero, and thus no power would cross the interface. This result is in accord with the statement that these fields resemble those on the reverse side of a boundary supporting total internal reflection.

4. It is a *plane wave* because its spatial variation takes the form $e^{-\vec{\gamma}\cdot\ \mathbf{r}}$, and thus there are planes of constant phase, these being in the direction perpendicular to $\vec{\beta}$, but is not a *uniform plane wave* as α is not zero.

L.22 More Transmision Line Interpretation of Wave Reflection

A uniform plane transverse electromagnetic wave at a frequency of 3750 MHz is normally incident from the left upon a lossless dielectric slab, of thickness 10 mm and relative dielectric constant 4, which is backed by a perfectly conducting plate in the plane $z = 0$. If the wave is polarised with its electric field parallel to the x-axis, find the following:

1. The transmission line arrangement analogous to the slab and plate assembly.
2. The resultant reflection co-efficient at the air-dielectric interface.
3. Expressions for the resultant electric and magnetic field distributions in the free space region to the left of the air-dielectric interface.
4. Expressions for the resultant electric and magnetic field distributions in the dielectric.
5. Using the above results or otherwise determine the percentage of the incident power which is reflected by the structure.

Answer

1. The transmission line arrangement analogous to the assembly is shown in Figure L.28.

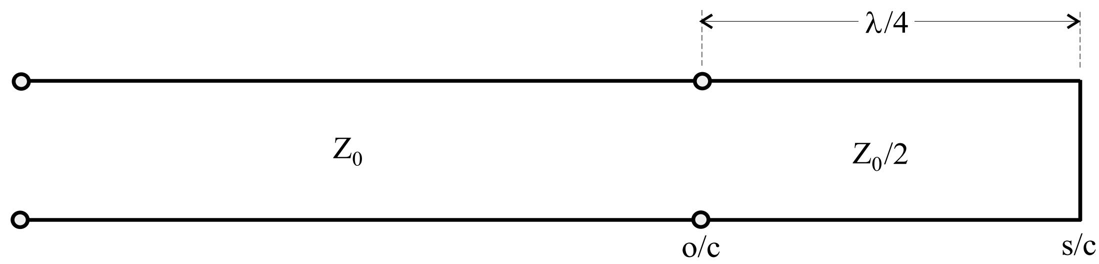


Figure L.28: Transmission line representation of slab against metallic plate.

2. As the slab acts as a shorted quarter wave of line, the slab will appear as an open circuit at the air-dielectric interface, and has there a reflection coefficeint of 1.
3. In the free space region to the left of the air-dielectric interface, there will be an incident wave and an equal magnitude reflected wave. It is convenient to place the origin of the z axis in the plane of the air dielectric interface. The incident, reflected and total electric fields will then be

$$\mathbf{E}^{a,i}(x,y,z) = \begin{bmatrix} \mathrm{E}_a \\ 0 \\ 0 \end{bmatrix} e^{-j\beta_{\mathrm{a}} z} \tag{L.56}$$

$$\mathbf{E}^{a,r}(x,y,z) = \begin{bmatrix} \mathrm{E}_a \\ 0 \\ 0 \end{bmatrix} e^{j\beta_\mathrm{a} z} \tag{L.57}$$

$$\mathbf{E}^{a,t}(x,y,z) = 2 \begin{bmatrix} \mathrm{E}_a \\ 0 \\ 0 \end{bmatrix} \cos(\beta_a z) \tag{L.58}$$

4. Now in the dielectric here will be complete reflection at the short circuiting plate. If we have in the dielectric a forward wave

$$\mathbf{E}^{d,i}(x,y,z) = \begin{bmatrix} \mathrm{E}_d \\ 0 \\ 0 \end{bmatrix} e^{-\beta_\mathrm{d} z} \tag{L.59}$$

then the reverse wave will be

$$\mathbf{E}^{d,r}(x,y,z) = \begin{bmatrix} \mathrm{E}_d \\ 0 \\ 0 \end{bmatrix} e^{\beta_\mathrm{d} z} \tag{L.60}$$

while the total wave in the the dielectric will be

$$\mathbf{E}^{d,t}(x,y,z) = 2 \begin{bmatrix} \mathrm{E}_d \\ 0 \\ 0 \end{bmatrix} \cos(\beta_d z) \tag{L.61}$$

Matching the electric fields at the air-dielectric interface gives 2 $\mathrm{E}_d = 2\ \mathrm{E}_a$. Thus

$$\mathbf{E}^{d,t}(x,y,z) = 2 \begin{bmatrix} \mathrm{E}_a \\ 0 \\ 0 \end{bmatrix} \cos(\beta_d z) \tag{L.62}$$

The corresponding magnetic fields are obtained from the above forward and backward waves and are

$$\mathbf{H}^{d,i}(x,y,z) = \begin{bmatrix} 0 \\ \mathrm{E}_a/\eta_d \\ 0 \end{bmatrix} e^{-j\beta_\mathrm{d} z} \tag{L.63}$$

$$\mathbf{H}^{d,r}(x,y,z) = \begin{bmatrix} 0 \\ -\mathrm{E}_{a/\eta_\mathrm{d}} \\ 0 \end{bmatrix} e^{\beta_\mathrm{d} z} \tag{L.64}$$

$$\mathbf{H}^{d,t}(x,y,z) = 2 \begin{bmatrix} 0 \\ \mathrm{E}_a/\eta_d \\ 0 \end{bmatrix} 2j\sin(\beta_d z) \tag{L.65}$$

5. It is clear that, as neither the dielectric nor the short circuiting plate has any mechanism for energy dissipation, all of the incident power will be reflected by the structure. This result will hold for all frequencies, not only the frequency for which the slab is a quarter wave in thickness.

L.23 Antennas

1. Determine the power receivd by a properly matched antenna which is distant 1000 m from a transmitter antenna radiating a power of 100 W under the following conditions:

   (a) Gain of transmitting antenna = 1.5, effective area of receiver = 0.4 m^2.

   (b) Gain of both antennas = 2, wavelength = 0.1 m.

   (c) Effective area of both antennas = 1 m^2, wavelength = 0.03 m.

2. A small square loop antenna of side L centered at the origin and oriented with sides parallel to the x and y axes carries a sinusoidal current of which the phasor I represents the peak value.

   (a) Calculate the radiation vector $\mathbf{R}(\theta, 0)$ as a function of θ for directions in the x, z plane.

   (b) Locate the direction of strongest radiation.

   (c) Obtain expressions for the electric and magnetic field components in this direction.

Answer

1. Using the formulae

$$\frac{P_r}{P_t} = \left(\frac{g_t}{4\pi r^2}\right) A_{er} \tag{L.66}$$

$$\frac{P_r}{P_t} = g_r g_t \left(\frac{\lambda}{4\pi r}\right)^2 \tag{L.67}$$

$$\frac{P_r}{P_t} = \frac{A_{er} A_{et}}{\lambda^2 r^2} \tag{L.68}$$

we derive the results

   (a) 4.775 μW.

   (b) 25.3 nW.

   (c) 111 mW.

2. (a) In the xz plane only the currents along the sides of the loop in the y direction will make a contribution to the vector potential. The contributions from the sides along the x direction will cancel, as for each element there is an eqidistant element of equal size carrying current in the opposie diection.

   The two sides parallel to the y axis will make a contribution to only the y (or ϕ) component of $\mathbf{R}$ or $\mathbf{A}$. The total contribution to the ϕ component of the radiation vector $\mathbf{R}(\theta, 0)$ is thus

$$\mathrm{R}_\phi(\theta,0) = j\beta\ \mathrm{I}L^2 \sin\theta \tag{L.69}$$

   (b) The direction of strongest direction is $\theta = \pi/2$.

   (c) Using the results

$$\begin{aligned}\mathrm{H}_\theta &= j\beta \frac{e^{-j\beta r}}{4\pi r} R_\phi \\ \mathrm{H}_\phi &= -j\beta \frac{e^{-j\beta r}}{4\pi r} R_\theta\end{aligned}$$

   and the familiar equation

$$\mathrm{E}_\phi = -\eta\ \mathrm{H}_\theta \tag{L.70}$$

   we obtain the results

$$\begin{aligned}\mathrm{H}_\theta &= \frac{j\beta e^{-j\beta r}}{4\pi r}\ \mathbf{R}_\phi = \frac{-(\beta L)^2\ \mathrm{I}\sin\theta}{4\pi r} e^{-j\beta r} \\ \mathrm{E}_\phi &= -\eta\ \mathrm{H}_\theta = \frac{(\beta L)^2 \eta\ \mathrm{I}\sin\theta}{4\pi r} e^{-j\beta r}\end{aligned}$$

Let us give grateful thanks for having come to the end of this tribulation.